Parametricity and syntactic logical relations in System F

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Abstract

We give a simple syntactic proof of a parametricity theorem for the polymorphic lambda calculus. As an application, we prove confluence and normalisation. We also indicate how to use this parametricity result to derive Wadler-style "free theorems".

1 Introduction

Reynolds [5] proved the parametricity theorem for the polymorphic lambda calculus, which essentially states that every term in System F satisfies a suitable notion of logical relation. Most presentations of the parametricity theorem are formulated semantically — they refer to specific classes of models [5, 8, 4, 9]. We provide a syntactic treatment of the parametricity theorem. In fact, our treatment can also be seen as implicitly referring to a specific kind of semantics constructed from the term model. The parametricity theorem may then be seen as a soundness theorem for this implicit semantics.

The syntactic treatment allows us to use the parametricity theorem to derive what we call an admissibility theorem: a generalised version of Girard's method of candidates. This theorem may in turn be used to give simple proofs of, e.g., confluence and strong normalisation of $\beta\eta$-reduction in System F.

In the context of the simply typed lambda calculus, logical relations were introduced by Statman [7]. The notion of syntactic logical relations for the simply typed lambda calculus is well-established [1, Section 3.3], and in fact already appeared in [7]. We extend the notion of syntactic logical relations to System F. From this point of view, the fundamental theorem for syntactic logical relations (see e.g. [1, Theorem 3.3.12]) corresponds to the parametricity theorem in our treatment. In the simply typed setting, the fundamental theorem may be used to show confluence and weak normalisation of $\beta\eta$-reduction.

The parametricity theorem has been used by Wadler [8] to derive "free theorems" from the types of terms in System F. In [8] these theorems refer to equality in frame models. The syntactic version of the parametricity theorem allows us to derive such free theorems with $\beta\eta$-equality instead.

Gallier [2] provides a generalisation of Girard's reducibility cadidates very similar to our syntactic logical relations. Our Parametricity Theorem 3.7 is analogous to [2, Lemma 7.9] and our Admissibility Theorem 4.5 to [2, Theorem 10.1]. Gallier uses generalised candidates of reducibility to show confluence and strong normalisation of well-typed System F terms [2, Lemma 10.2]. He considers only unary relations and does not use his method to derive free theorems. The present note may be seen as a small generalisation and a streamlined presentation of the results of [2].

2 Polymorphic lambda calculus

In this section, we define an orthodox Church-style version of System $\mathtt{F}$. See e.g. [6, Chapter 11] or [3, Chapter 11]. We assume familiarity with core notions of lambda calculi such as substitution and α-conversion.

Definition 2.1. *Types* $\mathcal{T}$ *are given by*

$$\mathcal{T} \quad ::= \quad \mathcal{V} \mid \mathcal{T} \to \mathcal{T} \mid \forall \alpha . \mathcal{T}$$

where $\mathcal{V}$ *is an infinite set of type variables.*

We define $\mathrm{FTV}(\tau)$ – the set of free type variables of the type τ – in an obvious way by induction on τ. A type τ is *closed* if $\mathrm{FTV}(\tau) = \emptyset$.

Definition 2.2. We assume given an infinite set Vars of variables, each paired with a unique type, denoted $x : \tau$.

The set of *terms* consists of all expressions s such that $s : \sigma$ can be inferred for some type σ by the following clauses:

- $x : \sigma$ for $(x : \sigma) \in \mathrm{Vars}$,
- $\lambda x : \sigma . s : \sigma \to \tau$ if $(x : \sigma) \in \mathrm{Vars}$ and $s : \tau$,
- $\Lambda \alpha . s : \forall \alpha . \sigma$ if $s : \sigma$ and α does not occur free in the type of a free variable of s,
- $st : \tau$ if $s : \sigma \to \tau$ and $t : \sigma$,
- $s\tau : \sigma[\tau/\alpha]$ if $s : \forall \alpha . \sigma$ and τ is a type.

The set of free variables of a preterm t, denoted $\mathrm{FV}(t)$, is defined in the expected way. Analogously, we define the set $\mathrm{FTV}(t)$ of type variables occurring free in t (we include the occurrences in the types of free variables). We denote an occurrence of a variable x of type τ by x^τ, e.g. $\lambda x : \tau \to \sigma . x^{\tau \to \sigma} y^\tau$. When clear or irrelevant, we omit the type annotations, denoting the above term by $\lambda x . xy$. Type substitution is defined in the expected way except that it needs to change the types of variables. Formally, a type substitution changes the types associated to variables in Vars. The set of terms of type τ is denoted by $\mathbb{T}_\tau$.

Note that we present terms in orthodox Church-style, i.e., instead of using contexts each variable has a globally fixed type associated to it.

Lemma 2.3 (Substitution lemma)**.** *1. If $s : \tau$ and $x : \sigma$ and $t : \sigma$ then $s[t/x] : \tau$.*

2. If $t : \sigma$ then $t[\tau/\alpha] : \sigma[\tau/\alpha]$.

Proof. Induction on the typing derivation. □

Lemma 2.4 (Generation lemma)**.** *If $t : \sigma$ then one of the following holds.*

- *$t \equiv x$ is a variable with $(x : \sigma) \in \mathrm{Vars}$.*
- *$t \equiv \lambda x : \tau_1 . s$ and $\sigma = \tau_1 \to \tau_2$ and $s : \tau_2$.*
- *$t \equiv \Lambda \alpha . s$ and $\sigma = \forall \alpha . \tau$ and $s : \tau$ and α does not occur free in the type of a free variable of s.*
- *$t \equiv t_1 t_2$ and $t_1 : \tau \to \sigma$ and $t_2 : \tau$ and $\mathrm{FTV}(\tau) \subseteq \mathrm{FTV}(t)$.*
- *$t \equiv s\tau$ and $\sigma = \rho[\tau/\alpha]$ and $s : \forall \alpha . \rho$.*

Proof. By analysing the derivation $t : \sigma$. □

3 Parametricity and logical relations

Definition 3.1. A relation R on $\mathbb{T}_{\tau_1} \times \ldots \times \mathbb{T}_{\tau_n}$ has *type* $(\tau_1, \ldots, \tau_n)$. For a family $\mathtt{Rel}$ of n-ary relations, by $\mathtt{Rel}_{\tau_1,\ldots,\tau_n}$ we denote the relations in $\mathtt{Rel}$ of type $(\tau_1, \ldots, \tau_n)$.

Given R of type $(\sigma_1, \ldots, \sigma_n)$ and S of type $(\tau_1, \ldots, \tau_n)$, we define the relation $R \to S$ of type $(\sigma_1 \to \tau_1, \ldots, \sigma_n \to \tau_n)$ by:

- $(R \to S)(t_1, \ldots, t_n)$ iff for all $s_1, \ldots, s_n$ with $R(s_1, \ldots, s_n)$ we have $S(t_1 s_1, \ldots, t_n s_n)$.

Given $\tau_1, \ldots, \tau_n$ and a family $\mathcal{F}$ of n-ary relations, we define $\forall\mathcal{F}$ of type $(\forall\alpha\tau_1, \ldots, \forall\alpha\tau_n)$ by:

- $(\forall\mathcal{F})(t_1, \ldots, t_n)$ iff for all types $\sigma_1, \ldots, \sigma_n$ and all $R \in \mathcal{F}$ of type $(\tau_1[\sigma_1/\alpha], \ldots, \tau_n[\sigma_n/\alpha])$ we have $R(t_1\sigma_1, \ldots, t_n\sigma_n)$.

Let S be an n-ary relation on terms. A relation R of type $(\tau_1, \ldots, \tau_n)$ is *closed under S-compatible head β-expansion* if the following properties hold:

- if $R(u^1[w_1^1/x]w_2^1 \ldots w_k^1, \ldots, u^n[w_1^n/x]w_2^n \ldots w_k^n)$ and for all $i = 1, \ldots, k$ either all w_i^j are types or $S(w_i^1, \ldots, w_i^n)$, then $R((\lambda x.u^1)w_1^1 \ldots w_k^1, \ldots, (\lambda x.u^n)w_1^n \ldots w_k^n)$;
- if $R(u^1[\tau/\alpha]w_1^1 \ldots w_k^1, \ldots, u^n[\tau/\alpha]w_1^n \ldots w_k^n)$ and for all $i = 1, \ldots, k$ either all w_i^j are types or $S(w_i^1, \ldots, w_i^n)$, then $R((\Lambda\alpha.u^1)\tau w_1^1 \ldots w_k^1, \ldots, (\Lambda\alpha.u^n)\tau w_1^n \ldots w_k^n)$.

A relation is *closed under head β-expansion* if it is closed under S-compatible head β-expansion for any relation S. Given a family $\mathtt{Rel}$ of n-ary relations, a relation is *closed under $\mathtt{Rel}$-compatible head β-expansion* if it is closed under S-compatible head β-expansion for every $S \in \mathtt{Rel}$.

A family $\mathtt{Rel}$ of n-ary relations is a *family of logical relations* if it satisfies the following:

1. each $R \in \mathtt{Rel}_{\tau_1,\ldots,\tau_n}$ is closed under $\mathtt{Rel}$-compatible head β-expansion;
2. $\mathtt{Rel}_{\alpha,\ldots,\alpha} \neq \emptyset$ for each type variable α;
3. if $R \in \mathtt{Rel}_{\sigma_1,\ldots,\sigma_n}$ and $S \in \mathtt{Rel}_{\tau_1,\ldots,\tau_n}$ then $R \to S \in \mathtt{Rel}_{\sigma_1\to\tau_1,\ldots,\sigma_n\to\tau_n}$;
4. if $\mathcal{F} \subseteq \mathtt{Rel}$ and $\mathcal{F}_{\tau_1,\ldots,\tau_n} \neq \emptyset$ then $\forall\mathcal{F} \in \mathtt{Rel}_{\forall\alpha\tau_1,\ldots,\forall\alpha\tau_n}$.

For the rest of this section, we fix a family of logical relations $\mathtt{Rel}$.

Definition 3.2. An n-mapping ω is a mapping from type variables to n-tuples of types. The mapping ω extends in an obvious way to a mapping from types to n-tuples of types. We set $\omega_i = \pi_i \circ \omega$, i.e., $\omega_i(\tau)$ is the i-th component of the tuple $\omega(\tau)$. A mapping ξ on type variables is *ω-compatible* if $\xi(\alpha) \in \mathtt{Rel}_{\omega_1(\alpha),\ldots,\omega_n(\alpha)}$.

For each type σ, each n-mapping ω, and each ω-compatible ξ, we define the n-ary relation $\mathcal{R}_\sigma^{\xi,\omega}$ by induction on σ:

- $\mathcal{R}_\alpha^{\xi,\omega} = \xi(\alpha)$ for a type variable α,
- $\mathcal{R}_{\sigma\to\tau}^{\xi,\omega}(t_1, \ldots, t_n)$ iff $t_i : \omega_i(\sigma \to \tau)$ and for all $s_1, \ldots, s_n$ such that $\mathcal{R}_\sigma^{\xi,\omega}(s_1, \ldots, s_n)$ we have $\mathcal{R}_\tau^{\xi,\omega}(t_1 s_1, \ldots, t_n s_n)$,
- $\mathcal{R}_{\forall\alpha\sigma}^{\xi,\omega}(t_1, \ldots, t_n)$ iff $t_i : \omega_i(\forall\alpha\sigma)$ and for all types $\tau_1, \ldots, \tau_n$ and every $R \in \mathtt{Rel}_{\tau_1',\ldots,\tau_n'}$ we have $\mathcal{R}_\sigma^{\xi',\omega'}(t_1\tau_1', \ldots, t_n\tau_n')$ where $\tau_i' = \omega_i(\tau_i)$ and $\xi' = \xi[R/\alpha]$ and $\omega' = \omega[(\tau_1', \ldots, \tau_n')/\alpha]$.

Note that if $\mathcal{R}_\sigma^{\xi,\omega}(t_1, \ldots, t_n)$ then $t_i : \omega_i(\sigma)$.

Lemma 3.3. *If ω is an n-mapping and ξ is ω-compatible, then $\mathcal{R}_\tau^{\xi,\omega} \in \mathtt{Rel}_{\omega_1(\tau),\ldots,\omega_n(\tau)}$.*

Proof. Induction on τ, using the properties of a family of logical relations. □

Lemma 3.4. *If ω is an n-mapping and ξ is ω-compatible and $\omega_i(\alpha) = \alpha$, then $\mathcal{R}^{\xi,\omega}_{\sigma[\tau/\alpha]} = \mathcal{R}^{\xi',\omega'}_{\sigma}$ where $\xi' = \xi[\mathcal{R}^{\xi,\omega}_{\tau}/\alpha]$ and $\omega' = \omega[(\omega_1(\tau),\ldots,\omega_n(\tau))/\alpha]$.*

Proof. Induction on σ. □

Definition 3.5. A *replacement* is a function $\delta = \gamma \circ \omega$ satisfying:

1. ω is a type substitution,
2. γ is a term substitution such that $\gamma(x^\tau) : \omega(\tau)$ for every variable x.

For τ a type, we use $\delta(\tau)$ to denote $\omega(\tau)$. We use the notation $\delta[t/x] = \gamma[t/x] \circ \omega$. Note that if $t : \tau$ then $\delta(t) : \delta(\tau)$.

Lemma 3.6. *If $t : \sigma$ and $\delta_i = \gamma_i \circ \omega_i$ for $i = 1,\ldots,n$ are replacements such that $\mathcal{R}^{\xi,\omega}_{\tau}(\delta_1(x),\ldots,\delta_n(x))$ for $x^\tau \in \mathrm{FV}(t)$, then $\mathcal{R}^{\xi,\omega}_{\sigma}(\delta_1(t),\ldots,\delta_n(t))$.*

Proof. Induction on t. If $t = x$ then this follows from the assumption.

If $t = t_1 t_2$ then $t_1 : \tau \to \sigma$ and $t_2 : \tau$. By the inductive hypothesis $\mathcal{R}^{\xi,\omega}_{\tau\to\sigma}(\delta_1(t_1),\ldots,\delta_n(t_1))$ and $\mathcal{R}^{\xi,\omega}_{\tau}(\delta_1(t_2),\ldots,\delta_n(t_2))$. By the definition of $\mathcal{R}^{\xi,\omega}_{\tau\to\sigma}$ we have $\mathcal{R}^{\xi,\omega}_{\sigma}(\delta_1(t_1 t_2),\ldots,\delta_n(t_1 t_2))$, i.e., $\mathcal{R}^{\xi,\omega}_{\sigma}(\delta_1(t),\ldots,\delta_n(t))$.

If $t = \lambda x : \sigma_1.u$ then $u : \sigma_2$ and $\sigma = \sigma_1 \to \sigma_2$. Let $s_1,\ldots,s_n$ be such that $\mathcal{R}^{\xi,\omega}_{\sigma_1}(s_1,\ldots,s_n)$. Let $\delta'_i = \delta_i[s_i/x]$ for $i = 1,\ldots,n$. This is well-defined, because $s_i : \omega_i(\sigma_1)$ for $i = 1,\ldots,n$. Also, δ'_i still satisfy the assumption of the theorem. Hence, by the inductive hypothesis $\mathcal{R}^{\xi,\omega}_{\sigma_2}(\delta'_1(u),\ldots,\delta'_n(u))$. We have $\delta_i(\lambda x : \sigma_1.u)s_i \to_{h\beta} \delta_i(u)[s_i/x] = \delta'_i(u)$ (assuming x is chosen fresh). Since $\mathcal{R}^{\xi,\omega}_{\sigma_1}(s_1,\ldots,s_n)$, by Lemma 3.3 and property 1 of a family of logical relations we obtain $\mathcal{R}^{\xi,\omega}_{\sigma_2}(\delta_1(t)s_1,\ldots,\delta_n(t)s_n)$. This proves $\mathcal{R}^{\xi,\omega}_{\sigma}(\delta_1(t),\ldots,\delta_n(t))$.

If $t = s\rho$ then $s : \forall\alpha\tau$ and $\sigma = \tau[\rho/\alpha]$. By the inductive hypothesis $\mathcal{R}^{\xi,\omega}_{\forall\alpha\tau}(\delta_1(s),\ldots,\delta_n(s))$. By Lemma 3.3 we have $\mathcal{R}^{\xi,\omega}_{\rho} \in \mathtt{Rel}_{\omega_1(\rho),\ldots,\omega_n(\rho)}$, so $\mathcal{R}^{\xi',\omega'}_{\tau}(\delta_1(t),\ldots,\delta_n(t))$ by definition, where $\xi' = \xi[\mathcal{R}^{\xi,\omega}_{\rho}/\alpha]$ and $\omega' = \omega[(\omega_1(\rho),\ldots,\omega_n(\rho))/\alpha]$. By Lemma 3.4 (assuming α chosen fresh) we obtain $\mathcal{R}^{\xi,\omega}_{\tau[\rho/\alpha]}(\delta_1(t),\ldots,\delta_n(t))$, i.e., $\mathcal{R}^{\xi,\omega}_{\sigma}(\delta_1(t),\ldots,\delta_n(t))$.

If $t = \Lambda\alpha.s$ then $s : \tau$ and $\sigma = \forall\alpha\sigma'$. Let $\rho_1,\ldots,\rho_n$ be types and let $R \in \mathtt{Rel}_{\omega_1(\rho_1),\ldots,\omega_n(\rho_n)}$. Let $\rho'_i = \omega_i(\rho_i)$ and $\xi' = \xi[R/\alpha]$ and $\omega' = \omega[(\rho'_1,\ldots,\rho'_n)/\alpha]$. Let $\delta'_i = \gamma_i \circ \omega'_i$. Assuming α is chosen fresh, δ'_i is still a replacement, and $\mathcal{R}^{\xi',\omega'}_{\tau}(\delta'_1(x),\ldots,\delta'_n(x))$ for $x^\tau \in \mathrm{FV}(s)$. Hence by the inductive hypothesis $\mathcal{R}^{\xi',\omega'}_{\sigma'}(\delta'_1(s),\ldots,\delta'_n(s))$. Since $\delta_i(t)\rho'_i \to_{h\beta} \delta'_i(s)$, by Lemma 3.3 and property 1 of a family of logical relations we obtain $\mathcal{R}^{\xi',\omega'}_{\sigma'}(\delta_1(t)\rho_1,\ldots,\delta_n(t)\rho_n)$. This shows $\mathcal{R}^{\xi,\omega}_{\sigma}(\delta_1(t),\ldots,\delta_n(t))$. □

The parametricity theorem is a specialisation of the above lemma. We set $\mathcal{R}^{\xi}_{\tau} = \mathcal{R}^{\xi,\mathtt{id}}_{\tau}$ where $\mathtt{id}(\alpha) = (\alpha,\ldots,\alpha)$ for any type variable α.

Theorem 3.7 (Parametricity theorem)**.** *Let $\mathtt{Rel}$ be a family of logical relations and ξ a mapping such that $\xi(\alpha) \in \mathtt{Rel}_{\alpha,\ldots,\alpha}$ for each type variable α. If $t : \tau$ and for all $x^\sigma \in \mathrm{FV}(t)$ we have $\mathcal{R}^{\xi}_{\sigma}(x,\ldots,x)$, then $\mathcal{R}^{\xi}_{\tau}(t,\ldots,t)$ and $\mathcal{R}^{\xi}_{\tau} \in \mathtt{Rel}_{\tau,\ldots,\tau}$.*

Proof. We have $\mathcal{R}^{\xi}_{\tau}(t,\ldots,t)$ by Lemma 3.6. Also $\mathcal{R}^{\xi}_{\tau} \in \mathtt{Rel}$ by Lemma 3.3. □

4 Candidates

The parametricity theorem allows us to generalise Girard's method of candidates.

Definition 4.1. Let R be an n-ary relation on terms. A tuple $(xu_1^1 \dots u_m^1, \dots, xu_1^n \dots u_m^n)$ is R-*neutral* if for every $i = 1, \dots, m$ either all u_i^j are types or $R(u_i^1, \dots, u_i^n)$. For unary relations, we identify 1-tuples with their elements and talk about R-neutral terms.

An n-ary relation R is *admissible* if it satisfies the following:

1. R is closed under R-compatible head β-expansion;
2. $R(t_1, \dots, t_n)$ for every R-neutral tuple $(t_1, \dots, t_n)$;
3. if $R(t_1 x, \dots, t_n x)$ and $x \notin \mathrm{FV}(t_1, \dots, t_n)$ then $R(t_1, \dots, t_n)$;
4. if $R(t_1 \alpha, \dots, t_n \alpha)$ and $\alpha \notin \mathrm{FTV}(t_1, \dots, t_n)$ then $R(t_1, \dots, t_n)$.

We will show that if R is admissible then $R(t, \dots, t)$ holds for every term t. For this purpose, we define R-candidates and show that the family of all R-candidates is a family of logical relations.

Definition 4.2. A relation S of type $(\tau_1, \dots, \tau_n)$ is an R-*candidate* if:

1. $S \subseteq R$;
2. S is closed under R-compatible head β-expansion;
3. $S(t_1, \dots, t_n)$ for every R-neutral tuple $(t_1, \dots, t_n) \in \mathbb{T}_{\tau_1} \times \dots \times \mathbb{T}_{\tau_n}$.

Lemma 4.3. *Let R be admissible and let $R_{\tau_1, \dots, \tau_n} = R \cap (\mathbb{T}_{\tau_1} \times \dots \times \mathbb{T}_{\tau_n})$. Then $R_{\tau_1, \dots, \tau_n}$ is an R-candidate of type $(\tau_1, \dots, \tau_n)$.*

Proof. Follows directly from definitions. □

Lemma 4.4. *If R is admissible then the family $\mathtt{Cand}^R$ of all R-candidates is a family of logical relations.*

Proof. If S is an R-candidate then it is closed under R-compatible head β-expansion. Hence, for any $S' \in \mathtt{Cand}^R$, the relation S is closed under S'-compatible head β-expansion, because $S' \subseteq R$. Thus S is closed under $\mathtt{Cand}^R$-compatible head β-expansion.

By Lemma 4.3 we have $\mathtt{Cand}^R_{\alpha, \dots, \alpha} \neq \emptyset$.

Let $S_1 \in \mathtt{Cand}^R_{\sigma_1, \dots, \sigma_n}$ and $S_2 \in \mathtt{Cand}^R_{\tau_1, \dots, \tau_n}$. We need to show $S_1 \to S_2 \in \mathtt{Cand}^R_{\sigma_1 \to \tau_1, \dots, \sigma_n \to \tau_n}$. We check the properties of an R-candidate.

1. Let $(S_1 \to S_2)(t_1, \dots, t_n)$. Let $x \notin \mathrm{FV}(t_1, \dots, t_n)$. Because $(x, \dots, x)$ is R-neutral, $S_1(x, \dots, x)$. Then $S_2(t_1 x, \dots, t_n x)$, so $R(t_1 x, \dots, t_n x)$. Thus $R(t_1, \dots, t_n)$, because R is admissible.
2. $S_1 \to S_2$ is closed under R-compatible head β-expansion because S_2 is and $S_1 \subseteq R$.
3. Let $(t_1, \dots, t_n) \in \mathbb{T}_{\sigma_1 \to \tau_1} \times \dots \times \mathbb{T}_{\sigma_n \to \tau_n}$ be R-neutral. Assume $S_1(s_1, \dots, s_n)$. Because $S_1 \in \mathtt{Cand}^R$, we have $R(s_1, \dots, s_n)$. Hence $(t_1 s_1, \dots, t_n s_n) \in \mathbb{T}_{\tau_1} \times \dots \times \mathbb{T}_{\tau_n}$ is R-neutral. So $S_2(t_1 s_1, \dots, t_n s_n)$. This proves $(S_1 \to S_2)(t_1, \dots, t_n)$.

Let $\mathcal{F} \subseteq \mathtt{Cand}^R$ with $\mathcal{F}_{\tau_1, \dots, \tau_n} \neq \emptyset$. We need to show $\forall \mathcal{F} \in \mathtt{Cand}^R_{\forall \alpha \tau_1, \dots, \forall \alpha \tau_n}$. We check the properties of an R-candidate.

1. Let $(\forall \mathcal{F})(t_1, \dots, t_n)$. Let $S \in \mathcal{F}_{\tau_1, \dots, \tau_n}$. We have $S(t_1 \alpha, \dots, t_n \alpha)$ for α fresh, so $R(t_1 \alpha, \dots, t_n \alpha)$. Thus $R(t_1, \dots, t_n)$, because R is admissible.

2. $\forall\mathcal{F}$ is closed under R-compatible head β-expansion because each $S \in \mathcal{F}$ is.

3. Let $(t_1, \ldots, t_n) \in \mathbb{T}_{\forall\alpha\tau_1} \times \ldots \times \mathbb{T}_{\forall\alpha\tau_n}$ be R-neutral. Then $(t_1\sigma_1, \ldots, t_n\sigma_n) \in \mathbb{T}_{\tau_1[\sigma_1/\alpha]} \times \ldots \times \mathbb{T}_{\tau_n[\sigma_n/\alpha]}$ is R-neutral. So $S(t_1\sigma_1, \ldots, t_n\sigma_n)$ for all $S \in \mathcal{F}$ of type $(\tau_1[\sigma_1/\alpha], \ldots, \tau_n[\sigma_n/\alpha])$. This proves $(\forall\mathcal{F})(t_1, \ldots, t_n)$. □

Theorem 4.5 (Admissibility theorem)**.** *If R is admissible, then $R(t, \ldots, t)$ for any term t.*

Proof. Assume $t : \tau$. By Lemma 4.4 the family $\mathtt{Cand}^R$ is a family of logical relations. Let $\xi(\alpha) = R_{\alpha,\ldots,\alpha}$ for a type variable α. We have $\xi(\alpha) \in \mathtt{Cand}^R_{\alpha,\ldots,\alpha}$ by Lemma 4.3. For every $x^\sigma \in \mathrm{FV}(t)$ the tuple $(x, \ldots, x)$ is R-neutral, so $S(x, \ldots, x)$ for every $S \in \mathtt{Cand}^R_{\sigma,\ldots,\sigma}$. By Lemma 3.3 we have $\mathcal{R}^\xi_\sigma \in \mathtt{Cand}^R_{\sigma,\ldots,\sigma}$. Thus $\mathcal{R}^\xi_\sigma(x, \ldots, x)$. Therefore, by the parametricity theorem $(t, \ldots, t) \in \mathcal{R}^\xi_\tau \in \mathtt{Cand}^R$. Since $\mathcal{R}^\xi_\tau \subseteq R$ by property 1 of R-candidates, $R(t, \ldots, t)$. □

5 Applications

5.1 Confluence

Let $\mathtt{Con}_{\beta\eta}$ be the set of all terms whose all subterms are $\beta\eta$-confluent, i.e., $t \in \mathtt{Con}_{\beta\eta}$ iff for every subterm t' of t and all t_1, t_2 such that $t' \to^*_{\beta\eta} t_i$ $(i = 1, 2)$ there exists s with $t_i \to^*_{\beta\eta} s$ $(i = 1, 2)$. By the admissibility theorem, to prove $\beta\eta$-confluence of System $\mathtt{F}$ it suffices to show that $\mathtt{Con}_{\beta\eta}$ is admissible. The proof essentially reduces to the following lemma.

Lemma 5.1. *If $t \to_{h\beta} t_1$ and $t \to_{\beta\eta} t_2$ then there is s with $t_1 \to^*_{\beta\eta} s$ and $t_2 \to^=_{h\beta} s$.*

Proof. We have $t = (\lambda x.u)w_1 \ldots w_n$ and $t_1 = u[w_1/x]w_2 \ldots w_n$ $(n \geq 1)$. If the $\beta\eta$-reduction $t \to_{\beta\eta} t_2$ occurs inside one of $u, w_1, \ldots, w_n$ then the claim is obvious. Otherwise, either the reduction $t \to_{\beta\eta} t_2$ is the head β-reduction and $t_2 = t_1$, or $u = u'x$ with $x \notin \mathrm{FV}(u')$ and $t_2 = u'w_1 \ldots w_n$. In the second case, however, also $u[w_1/x] = u'w_1$, so we may take $s = t_1 = t_2$. □

Lemma 5.2. *$\mathtt{Con}_{\beta\eta}$ is admissible.*

Proof. We check the properties from Definition 4.1.

1. It follows from Lemma 5.1 that $\mathtt{Con}_{\beta\eta}$ is closed under $\mathtt{Con}_{\beta\eta}$-compatible head β-expansion. Indeed, assume $t_0 = u[w_1/x]w_2 \ldots w_n \in \mathtt{Con}_{\beta\eta}$ and $t'_0 = (\lambda x.u)w_1 \ldots w_n \to_{h\beta} t_0$ with $w_i \in \mathtt{Con}_{\beta\eta}$ for $i = 1, \ldots, n$. Let t' be a subterm of t'_0. If t' is a subterm of w_i for some $i = 1, \ldots, n$, then $t' \in \mathtt{Con}_{\beta\eta}$ and in particular t' is $\beta\eta$-confluent. If t' is a subterm of $\lambda x.u$ then $t' \in \mathtt{Con}_{\beta\eta}$ because $u \in \mathtt{Con}_{\beta\eta}$. Otherwise, there is a subterm t of t_0 such that that $t' \to_{h\beta} t$. Assume $t' \to^*_{\beta\eta} t'_i$ $(i = 1, 2)$. By Lemma 5.1 there are t_1, t_2 such that $t \to^*_{\beta\eta} t_i$ and $t'_i \to^=_{h\beta} t_i$ $(i = 1, 2)$. Since $t \in \mathtt{Con}_{\beta\eta}$, there is s with $t'_i \to^=_{h\beta} t_i \to^*_{\beta\eta} s$ $(i = 1, 2)$.
2. If $xu_1 \ldots u_n$ is $\mathtt{Con}_{\beta\eta}$-neutral then each u_i is either a type or $u_i \in \mathtt{Con}_{\beta\eta}$. So $xu_1 \ldots u_n \in \mathtt{Con}_{\beta\eta}$.
3. If $tx \in \mathtt{Con}_{\beta\eta}$ then $t \in \mathtt{Con}_{\beta\eta}$ because t is a subterm of tx.
4. If $t\alpha \in \mathtt{Con}_{\beta\eta}$ then $t \in \mathtt{Con}_{\beta\eta}$ because t is a subterm of $t\alpha$. □

Corollary 5.3. *System $\mathtt{F}$ is $\beta\eta$-confluent.*

An entirely analogous proof shows that System $\mathtt{F}$ is β-confluent.

5.2 Weak normalisation

Let $\mathrm{WN}_{\beta\eta}$ be the set of all terms weakly normalising w.r.t $\beta\eta$-reduction. By the admissibility theorem, to prove weak normalisation of $\beta\eta$-reduction in System $\mathtt{F}$ it suffices to show that $\mathrm{WN}_{\beta\eta}$ is admissible.

Lemma 5.4. $\mathrm{WN}_{\beta\eta}$ *is admissible.*

Proof. We check the properties from Definition 4.1.

1. It is obvious that $\mathrm{WN}_{\beta\eta}$ is closed under head β-expansion.
2. If $xu_1 \dots u_n$ is $\mathrm{WN}_{\beta\eta}$-neutral then each u_i is either a type or $u_i \in \mathrm{WN}_{\beta\eta}$. So $xu_1 \dots u_n \in \mathrm{WN}_{\beta\eta}$.
3. If $tx \in \mathrm{WN}_{\beta\eta}$ then there is s in $\beta\eta$-normal form such that $tx \to_{\beta\eta}^{*} s$. Thus either $s = s'x$ and $t \to_{\beta\eta}^{*} s'$, or $tx \to_{\beta\eta}^{*} (\lambda x.t')x \to_{\beta} t' \to_{\beta\eta}^{*} s$, i.e., $t \to_{\beta\eta}^{*} \lambda x.s$, or $tx \to_{\beta\eta}^{*} (\lambda x.t'x)x \to_{\eta} t'x \to_{\beta\eta}^{*} s$, i.e., also $t \to_{\beta\eta}^{*} \lambda x.s$. In both cases t has a $\beta\eta$-normal form.
4. The proof that $t\alpha \in \mathrm{WN}_{\beta\eta}$ implies $t \in \mathrm{WN}_{\beta\eta}$ is analogous to the point above. □

Corollary 5.5. *System* $\mathtt{F}$ *is weakly normalising w.r.t.* $\beta\eta$*-reduction.*

5.3 Strong normalisation

Strong normalisation is a bit more difficult than weak normalisation, but also follows relatively easily from the admissibility theorem. Let $\mathrm{SN}_{\beta\eta}$ be the set of all terms strongly normalising w.r.t. $\beta\eta$-reduction.

Lemma 5.6. $\mathrm{SN}_{\beta\eta}$ *is admissible.*

Proof. We check the properties from Definition 4.1.

1. We need to show that $\mathrm{SN}_{\beta\eta}$ is closed under $\mathrm{SN}_{\beta\eta}$-compatible head β-expansion. Assume $u[w_1/x]w_2 \dots w_k \in \mathrm{SN}_{\beta\eta}$ and $w_i \in \mathrm{SN}_{\beta\eta}$ for $i = 1, \dots, n$. Let $(\lambda x.u)w_1 \dots w_n = t_0 \to_{\beta\eta} t_1 \to_{\beta\eta} t_2 \to_{\beta\eta} \dots$ be an infinite reduction. There are three possibilities.
   - $t_i = (\lambda x.u^i)w_1^i \dots w_k^i$ for each i and there is an infnite reduction from u or one of $w_1, \dots, w_k$. This contradicts $u[w_1/x]w_2 \dots w_k \in \mathrm{SN}_{\beta\eta}$ or $w_1 \in \mathrm{SN}_{\beta\eta}$.
   - There is i with $t_i = (\lambda x.u^i)w_1^i \dots w_k^i$ and $t_{i+1} = u^i[w_1^i/x]w_2^i \dots w_k^i$, where $u \to_{\beta\eta}^{*} u^i$ and $w_j \to_{\beta\eta}^{*} w_j^i$. But then there is an infinite reduction $u[w_1/x]w_2 \dots w_k \to_{\beta\eta}^{*} t_{i+1} \to_{\beta\eta} t_{i+2} \to_{\beta\eta} \dots$ Contradiction.
   - There is i with $t_i = (\lambda x.u^i x)w_1^i \dots w_k^i$ and $t_{i+1} = u^i w_1^i w_2^i \dots w_k^i$, where $x \notin \mathrm{FV}(u^i)$ and $u \to_{\beta\eta}^{*} u^i x$ and $w_j \to_{\beta\eta}^{*} w_j^i$. But then there is an infinite reduction $u[w_1/x]w_2 \dots w_k \to_{\beta\eta}^{*} t_{i+1} \to_{\beta\eta} t_{i+2} \to_{\beta\eta} \dots$ Contradiction.

   Similarly, one shows that if $u[\tau/x]w_1 \dots w_k \in \mathrm{SN}_{\beta\eta}$ then $(\Lambda\alpha.u)\tau w_1 \dots w_n \in \mathrm{SN}_{\beta\eta}$.
2. If $xu_1 \dots u_n$ is $\mathrm{SN}_{\beta\eta}$-neutral then each u_i is either a type or $u_i \in \mathrm{SN}_{\beta\eta}$. So $xu_1 \dots u_n \in \mathrm{SN}_{\beta\eta}$ (an infinite reduction from $xu_1 \dots u_n$ implies an infinite reduction from one of u_i).
3. If $tx \in \mathrm{SN}_{\beta\eta}$ then $t \in \mathrm{SN}_{\beta\eta}$ in particular.
4. If $t\alpha \in \mathrm{SN}_{\beta\eta}$ then $t \in \mathrm{SN}_{\beta\eta}$ in particular. □

Corollary 5.7. *System* $\mathtt{F}$ *is strongly normalising w.r.t.* $\beta\eta$*-reduction.*

5.4 Theorems for free

Let $\texttt{Rel}^n$ be the family of all n-ary relations closed under $\beta\eta$-conversion, i.e., $R \in \texttt{Rel}^n$ iff $R(t_1, \ldots, t_n)$ and $t_i =_{\beta\eta} t'_i$ for $i = 1, \ldots, n$ imply $R(t'_1, \ldots, t'_n)$ (provided t'_i has the same type as t_i for $i = 1, \ldots, n$).

Lemma 5.8. $\texttt{Rel}^n$ *is a family of logical relations.*

Proof. We check the conditions from Definition 3.1. Obviously, $\texttt{Rel}^n$ is closed under $\texttt{Rel}^n$-compatible head β-expansion. Also $\texttt{Rel}_{\alpha,\ldots,\alpha} \neq \emptyset$, because e.g. the full relation is closed under $\beta\eta$-conversion. As for the remaining two points, one easily checks that the operations $\to$ and $\forall$ preserve the property of being closed under $\beta\eta$-conversion. $\square$

Now we can use the parametricity theorem to prove e.g. that any polymorphic function of type $\forall\alpha.\alpha \to \alpha$ is an identity.

Lemma 5.9. *If $f : \forall\alpha.\alpha \to \alpha$ is closed then $f =_{\beta\eta} \Lambda\alpha.\lambda x : \alpha.x$.*

Proof. Let $x : \alpha$. By the parametricity theorem for $\texttt{Rel}^1$ we obtain $\mathcal{R}_{\forall\alpha.\alpha\to\alpha}(f)$. Consider the relation $R = \{t : \alpha \mid t =_{\beta\eta} x\}$. We have $R \in \texttt{Rel}^1_\alpha$. Let $\xi(\alpha) = R$. Then $\mathcal{R}^\xi_{\alpha\to\alpha}(f\alpha)$. Also $\mathcal{R}^\xi_\alpha = \xi(\alpha) = R$, so $\mathcal{R}^\xi_\alpha(x)$. Thus $\mathcal{R}^\xi_\alpha(f\alpha x)$, i.e., $f\alpha x =_{\beta\eta} x$. Hence $f =_\eta \Lambda\alpha.\lambda x.f\alpha x =_{\beta\eta} \Lambda\alpha.\lambda x.x$. $\square$

Similarly, we characterise the type $\texttt{bool} = \forall\alpha.\alpha \to \alpha \to \alpha$ as consisting of two constructors $\texttt{true} = \Lambda\alpha.\lambda xy.x$ and $\texttt{false} = \Lambda\alpha.\lambda xy.y$.

Lemma 5.10. *If $f : \texttt{bool}$ is closed then $f =_{\beta\eta} \texttt{true}$ or $f =_{\beta\eta} \texttt{false}$.*

Proof. By the parametricity theorem for $\texttt{Rel}^1$ we have $\mathcal{R}_{\texttt{bool}}(f)$. Let x, y be distinct variables of type α and let $\xi(\alpha) = \{t : \alpha \mid t =_{\beta\eta} x \vee t =_{\beta\eta} y\}$. Then $\xi(\alpha) \in \texttt{Rel}^1_\alpha$ and thus $\mathcal{R}^\xi_{\alpha\to\alpha\to\alpha}(f\alpha)$. Obviously, $\xi(\alpha)(x)$ and $\xi(\alpha)(y)$, so $\mathcal{R}^\xi_\alpha(f\alpha xy)$, i.e., $f\alpha xy =_{\beta\eta} x$ or $f\alpha xy =_{\beta\eta} y$. This implies $f =_{\beta\eta} \texttt{true}$ or $f =_{\beta\eta} \texttt{false}$. $\square$

The previous two lemmas could be proved with β- instead of $\beta\eta$-conversion, by analysing the normal forms of f, but this would depend on normalisation. The next lemma follows from the lemma above, but for illustrative purposes we give a direct proof that makes a more sophisticated use of the parametricity theorem for binary relations.

Lemma 5.11. *If $f : \texttt{bool}$ is closed and $g : \tau \to \sigma$ and $t_1, t_2 : \tau$, then $f\sigma(gt_1)(gt_2) =_{\beta\eta} g(f\tau t_1 t_2)$*

Proof. By the parametricity theorem for $\texttt{Rel}^2$ we have $\mathcal{R}_{\texttt{bool}}(f, f)$. Let $R = \{(s_1, s_2) \mid gs_1 =_{\beta\eta} s_2\}$. We have $R \in \texttt{Rel}^2_{\tau,\sigma}$ and $R(t_1, gt_1)$ and $R(t_2, gt_2)$. Let $\xi(\alpha) = R$ and $\omega(\alpha) = (\tau, \sigma)$. Then $\mathcal{R}^{\xi,\omega}_\alpha(f\tau t_1 t_2, f\sigma(gt_1)(gt_2))$, i.e., $g(f\tau t_1 t_2) =_{\beta\eta} f\sigma(gt_1)(gt_2)$. $\square$

Now we show that any polymorphic function into $\texttt{bool}$ is constant. First, we characterise the binary $\mathcal{R}^\xi_{\texttt{bool}}$.

Lemma 5.12. *If $\mathcal{R}^\xi_{\texttt{bool}}(t_1, t_2)$ then $t_1 =_{\beta\eta} t_2$.*

Proof. Let $R = \{(s_1, s_2) \mid s_1 =_{\beta\eta} s_2 \wedge s_1, s_2 : \alpha\}$. We have $R \in \texttt{Rel}^2_{\alpha,\alpha}$. Let $\xi(\alpha) = R$. Then $\mathcal{R}^\xi_{\alpha\to\alpha\to\alpha}(t_1\alpha, t_2\alpha)$. Since $\mathcal{R}^\xi_\alpha(x, x)$ for any variable $x : \alpha$, we obtain $\mathcal{R}^\xi_\alpha(t_1\alpha xy, t_2\alpha xy)$, i.e., $t_1\alpha xy =_{\beta\eta} t_2\alpha xy$, for $x, y \notin \mathrm{FV}(t_1, t_2)$. This implies $t_1 =_{\beta\eta} t_2$. $\square$

Lemma 5.13. *If $f : \forall\alpha.\alpha \to \mathtt{bool}$ is closed then for all types τ_1, τ_2 and terms $t_1 : \tau_1$, $t_2 : \tau_2$ we have $f\tau_1 t_1 =_{\beta\eta} f\tau_2 t_2$.*

Proof. By the parametricity theorem for $\mathtt{Rel}^2$ we have $\mathcal{R}_{\forall\alpha.\alpha\to\mathtt{bool}}(f, f)$. Let $R = \{(s_1, s_2) \mid s_1 : \tau_1, s_2 : \tau_2\}$. We have $R \in \mathtt{Rel}^2_{\tau_1,\tau_2}$. Let $\xi(\alpha) = R$ and $\omega(\alpha) = (\tau_1, \tau_2)$. Then $\mathcal{R}^{\xi,\omega}_{\mathtt{bool}}(f\tau_1 t_1, f\tau_2 t_2)$, because $\mathcal{R}^{\xi,\omega}_{\alpha}(t_1, t_2)$. By Lemma 5.12 we obtain $f\tau_1 t_1 =_{\beta\eta} f\tau_2 t_2$. □

Next, we consider lists, encoded impredicatively in System $\mathtt{F}$.

Definition 5.14. We define $\mathtt{List}(\tau) = \forall\alpha.(\tau \to \alpha \to \alpha) \to \alpha \to \alpha$. We use the abbreviation $[a_1, \ldots, a_n]$ for $\Lambda\alpha\lambda f x.f a_1(f a_2(\ldots(f a_n x)))$. In particular, $[] = \Lambda\alpha\lambda f x.x$. We use $a :: l$ for $\Lambda\alpha\lambda f x.f a(l\alpha f x)$.

Lemma 5.15. *If $l : \mathtt{List}(\tau)$ is closed then $l =_{\beta\eta} [a_1, \ldots, a_n]$ for some $a_1, \ldots, a_n : \tau$.*

Proof. By the parametricity theorem for $\mathtt{Rel}^1$ we have $\mathcal{R}_{\mathtt{List}(\tau)}(l)$. Given $x : \alpha$ and $f : \alpha \to \alpha \to \alpha$, define $R \in \mathtt{Rel}^1_\tau$ by: $R(t)$ iff $t : \tau$ and $t =_{\beta\eta} f a_1(f a_2(\ldots(f a_n x)))$ for some $a_1, \ldots, a_n : \tau$ (possibly $n = 0$). Let $\xi(\alpha) = R$. Let $f : \tau \to \alpha \to \alpha$ and $x : \alpha$ be variables.

We first show $\mathcal{R}^{\xi}_{\tau\to\alpha\to\alpha}(f)$. Let $a : \tau$ and $s : \alpha$ be such that $\mathcal{R}_\tau(a)$ and $\mathcal{R}^{\xi}_{\alpha}(s)$. Then $s =_{\beta\eta} f a_1(\ldots(f a_n x))$ for some $a_1, \ldots, a_n : \tau$. Hence $f a s =_{\beta\eta} f a(f a_1(\ldots(f a_n x)))$. This implies $\mathcal{R}^{\xi}_{\alpha}(f)$.

We also have $\mathcal{R}^{\xi}_{\alpha}(x)$. Thus $\mathcal{R}^{\xi}_{\alpha}(l\alpha f x)$. This implies our thesis. □

Lemma 5.16. *If ξ is ω-compatible and α is a type variable then $\mathcal{R}^{\xi,\omega}_{\mathtt{List}(\alpha)}([], [])$.*

Proof. Let τ_1, τ_2 be types, $R \in \mathtt{Rel}^2_{\tau_1,\tau_2}$, let β be a fresh type variable and let $\xi' = \xi[R/\beta]$ and $\omega' = \omega[(\tau_1, \tau_2)/\beta]$. Assume $\mathcal{R}^{\xi',\omega'}_{\alpha\to\beta\to\beta}(f_1, f_2)$ and $R(a_1, a_2)$. Since $[]\tau_1 f_1 a_1 =_{\beta\eta} a_1$ and $[]\tau_2 f_2 a_2 =_{\beta\eta} a_2$, and $\mathcal{R}^{\xi',\omega'}_{\beta} = R$ is closed under $\beta\eta$-conversion, we have $\mathcal{R}^{\xi',\omega'}_{\beta}([]\tau_1 f_1 a_1, []\tau_2 f_2 a_2)$. This implies $\mathcal{R}^{\xi,\omega}_{\mathtt{List}(\alpha)}([], [])$. □

Lemma 5.17. *If ξ is ω-compatible, α is a type variable, $\xi(\alpha) \neq \emptyset$, and*

$$\mathcal{R}^{\xi,\omega}_{\mathtt{List}(\alpha)}([a_1, \ldots, a_n], [b_1, \ldots, b_m])$$

then $n = m$ and $\mathcal{R}^{\xi,\omega}_{\alpha}(a_i, b_i)$ for every $i = 1, \ldots, n$.

Proof. Let ξ be ω-compatible and let $l_1 = [a_1, \ldots, a_n]$ and $l_2 = [b_1, \ldots, b_m]$. Assume $\omega(\alpha) = (\tau_1, \tau_2)$ and $\mathcal{R}^{\xi,\omega}_{\mathtt{List}(\alpha)}(l_1, l_2)$. We proceed by induction on n.

First assume $n, m > 0$. We have $\xi(\alpha) \in \mathtt{Rel}^2_{\tau_1,\tau_2}$, so $\mathcal{R}^{\xi,\omega}_{(\alpha\to\alpha\to\alpha)\to\alpha\to\alpha}(l_1\tau_1, l_2\tau_2)$. By Lemma 3.6 we obtain $\mathcal{R}^{\xi,\omega}_{\alpha\to\alpha\to\alpha}(\lambda x : \tau_1.\lambda y : \tau_1.x, \lambda x : \tau_2.\lambda y : \tau_2.x)$. Let $c : \tau_1$ and $d : \tau_2$ be such that $\xi(\alpha)(c, d)$. Then $\mathcal{R}^{\xi,\omega}_{\alpha}(l_1\tau_1(\lambda xy.x)c, l_2\tau_2(\lambda xy.x)d)$. We have $l_1\tau_1(\lambda xy.x)c =_{\beta\eta} a_1$ and $l_2\tau_2(\lambda xy.x)c =_{\beta\eta} b_1$. Since $\mathcal{R}^{\xi,\omega}_{\alpha} = \xi(\alpha) \in \mathtt{Rel}^2_{\tau_1,\tau_2}$, it is closed under $\beta\eta$-conversion. Thus $\mathcal{R}^{\xi,\omega}_{\alpha}(a_1, b_1)$.

Let β be a fresh type variable and let $\xi' = \xi[\mathcal{R}^{\xi,\omega}_{\mathtt{List}(\alpha)}/\beta]$ and $\omega' = \omega[(\mathtt{List}(\tau_1), \mathtt{List}(\tau_2))/\beta]$. Since $\xi'(\beta) \in \mathtt{Rel}^2_{\mathtt{List}(\tau_1),\mathtt{List}(\tau_2)}$ by Lemma 3.3, ξ' is ω'-comptabile, and $\mathcal{R}^{\xi',\omega'}_{(\alpha\to\beta\to\beta)\to\beta\to\beta}(l_1\tau_1, l_2\tau_2)$.

We have $\mathcal{R}^{\xi',\omega'}_{\alpha\to\beta\to\beta}(\lambda x:\tau_1.\lambda y:\mathtt{List}(\tau_1).y, \lambda x:\tau_2.\lambda y:\mathtt{List}(\tau_2).y)$ by Lemma 3.6. Also $\mathcal{R}^{\xi',\omega'}_{\beta}([],[])$ by Lemma 5.16 and Lemma 3.4. Hence $\mathcal{R}^{\xi',\omega'}_{\beta}(l_1\tau_1(\lambda xy.y)[], l_2\tau_2(\lambda xy.y)[])$, i.e.,

$$\mathcal{R}^{\xi,\omega}_{\mathtt{List}(\alpha)}(l_1\tau_1(\lambda xy.y)[], l_2\tau_2(\lambda xy.y)[]).$$

Because $l_1\tau_1(\lambda xy.y)[] =_{\beta\eta} [a_2,\ldots,a_n]$ and $l_2\tau_2(\lambda xy.y)[] =_{\beta\eta} [b_2,\ldots,b_m]$, and $\mathcal{R}^{\xi,\omega}_{\mathtt{List}(\alpha)}$ is closed under $\beta\eta$-conversion, we have $\mathcal{R}^{\xi,\omega}_{\mathtt{List}(\alpha)}([a_2,\ldots,a_n],[b_2,\ldots,b_m])$. By the inductive hypothesis $n=m$ and $\mathcal{R}^{\xi,\omega}_{\alpha}(a_i,b_i)$ for $i=2,\ldots,n$.

Now assume, e.g., $n=0$, i.e., $l_1=[]$. If $l_2=[]$ then we are done, so assume $l_2\neq[]$. Let β be a fresh type variable and define $R\in\mathtt{Rel}^2_{\beta,\beta}$ by $R(t_1,t_2)$ iff $t_1 =_{\beta\eta} t_2$. Let $a,b:\beta$ be non-$\beta\eta$-convertible. Let $\xi'=\xi[R/\beta]$ and $\omega'=\omega[R/\beta]$. We have $\mathcal{R}^{\xi',\omega'}_{\beta}(l_1\beta(\lambda xy.a)b, l_2\beta(\lambda xy.a)b)$, i.e., $l_1\beta(\lambda xy.a)b =_{\beta\eta} l_2\beta(\lambda xy.a)b$. But the left side is $\beta\eta$-convertible to a, while the right side is $\beta\eta$-convertible to b. Contradiction. □

Similarly, one can prove:

Lemma 5.18. *If ξ is ω-compatible, α is a type variable, and $\mathcal{R}^{\xi,\omega}_{\alpha}(a_i,b_i)$ for $i=1,\ldots,n$, then $\mathcal{R}^{\xi,\omega}_{\mathtt{List}(\alpha)}([a_1,\ldots,a_n],[b_1,\ldots,b_m])$.*

Combining the last three lemmas and Lemma 3.4, we obtain:

Corollary 5.19. *Assume ξ is ω-compatible and $\mathcal{R}^{\xi,\omega}_{\tau}\neq\emptyset$. Then $\mathcal{R}^{\xi,\omega}_{\mathtt{List}(\tau)}([a_1,\ldots,a_n],[b_1,\ldots,b_m])$ iff $n=m$ and $\mathcal{R}^{\xi,\omega}_{\tau}(a_i,b_i)$ for $i=1,\ldots,n$.*

Definition 5.20. We define $\mathtt{map} = \Lambda\alpha\beta.\lambda f:\alpha\to\beta.\lambda l:\mathtt{List}(\alpha).l(\mathtt{List}(\beta))(\lambda xy.fx :: y)$.

Lemma 5.21. $\mathtt{map}\,\tau\,\sigma\,f\,[a_1,\ldots,a_n] =_{\beta\eta} [fa_1,\ldots,fa_n]$.

Proof. By calculation. □

We can now show the free theorems from Wadler [8], with equality interpreted as $\beta\eta$-conversion.

Lemma 5.22. *If $r:\forall\alpha.\mathtt{List}(\alpha)\to\mathtt{List}(\alpha)$ is closed then for all τ_1,τ_2 and closed $f:\tau_1\to\tau_2$ and closed $l:\mathtt{List}(\tau_1)$ we have $\mathtt{map}\,\tau_1\,\tau_2\,f\,(r\tau_1 l) =_{\beta\eta} r\tau_2(\mathtt{map}\,\tau_1\,\tau_2\,f\,l)$.*

Proof. By the parametricity theorem we have $\mathcal{R}_{\forall\alpha.\mathtt{List}(\alpha)\to\mathtt{List}(\alpha)}(r,r)$. By Lemma 5.15 we have $l =_{\beta\eta} [a_1,\ldots,a_n]$. Let $R\in\mathtt{Rel}^2_{\tau_1,\tau_2}$ be defined by: $R(t_1,t_2)$ iff $ft_1 =_{\beta\eta} t_2$. Let $\xi(\alpha)=R$ and $\omega(\alpha)=(\tau_1,\tau_2)$. Then $\mathcal{R}^{\xi,\omega}_{\mathtt{List}(\alpha)\to\mathtt{List}(\alpha)}(r\tau_1,r\tau_2)$. We have $R(a_i,fa_i)$, i.e., $\mathcal{R}^{\xi,\omega}_{\alpha}(a_i,fa_i)$, for $i=1,\ldots,n$. Hence $\mathcal{R}^{\xi,\omega}_{\mathtt{List}(\alpha)}([a_1,\ldots,a_n],[fa_1,\ldots,fa_n])$ by Corollary 5.19. This implies $\mathcal{R}^{\xi,\omega}_{\mathtt{List}(\alpha)}(r\tau_1[a_1,\ldots,a_n], r\tau_2[fa_1,\ldots,fa_n])$, i.e., $\mathcal{R}^{\xi,\omega}_{\mathtt{List}(\alpha)}(r\tau_1 l, r\tau_2(\mathtt{map}\,\tau_1\,\tau_2\,f\,l))$, by Lemma 5.21 and closure under $\beta\eta$-conversion. By Lemma 5.15 we have $r\tau_1 l =_{\beta\eta} [b_1,\ldots,b_m]$ and $r\tau_2(\mathtt{map}\,\tau_1\,\tau_2\,f\,l) =_{\beta\eta} [b_1',\ldots,b_k']$. Thus $k=m$ and $b_i'=b_i$ for $i=1,\ldots,m$, by closure under $\beta\eta$-conversion and Corollary 5.19. So $r\tau_2(\mathtt{map}\,\tau_1\,\tau_2\,f\,l) =_{\beta\eta} [fb_1,\ldots,fb_m]$. By Lemma 5.21 this implies $\mathtt{map}\,\tau_1\,\tau_2\,f\,(r\tau_1 l) =_{\beta\eta} r\tau_2(\mathtt{map}\,\tau_1\,\tau_2\,f\,l)$. □

Definition 5.23. We define $\mathtt{fold} = \Lambda\alpha\beta.\lambda f:\alpha\to\beta\to\beta.\lambda a:\beta.\lambda l:\mathtt{List}(\alpha).l\beta f a$.

Lemma 5.24. *Let $f : \tau \to \sigma \to \sigma$ and $f' : \tau' \to \sigma' \to \sigma'$ be closed. Let $r_1 : \tau \to \tau'$ and $r_2 : \sigma \to \sigma'$ be closed and such that for all $t_1 : \tau$, $t_2 : \sigma$ we have $r_2(f t_1 t_2) =_{\beta\eta} f'(r_1 t_1)(r_2 t_2)$. Then for all $u : \sigma$ and all closed $l : \texttt{List}(\tau)$ we have $r_2(\texttt{fold}\,\tau\,\sigma\,f\,u\,l) =_{\beta\eta} \texttt{fold}\,\tau'\,\sigma'\,f'\,(r_2 u)(\texttt{map}\,\tau\,\tau'\,r_1 l)$.*

Proof. By the parametricity theorem $\mathcal{R}_{\forall\alpha\beta.(\alpha\to\beta\to\beta)\to\beta\to\texttt{List}(\alpha)\to\beta}(\texttt{fold}, \texttt{fold})$. Let $R_1 \in \texttt{Rel}^2_{\tau,\tau'}$ be defined by: $R_1(t_1, t_2)$ iff $r_1 t_1 =_{\beta\eta} t_2$. Analogously, define $R_2 \in \texttt{Rel}^2_{\sigma,\sigma'}$. Let $\xi(\alpha) = R_1$ and $\omega(\alpha) = (\tau, \tau')$ and $\xi(\beta) = R_2$ and $\omega(\beta) = (\sigma, \sigma')$. Then $\mathcal{R}^{\xi,\omega}_{(\alpha\to\beta\to\beta)\to\beta\to\texttt{List}(\alpha)\to\beta}(\texttt{fold}\,\tau\,\sigma, \texttt{fold}\,\tau'\,\sigma')$.

Next, we want to show that $\mathcal{R}^{\xi,\omega}_{\alpha\to\beta\to\beta}(f, f')$. This is equivalent to: for all $a : \tau$, $a' : \tau'$ with $R_1(a, a')$ and all $b : \sigma$, $b' : \sigma'$ with $R_2(b, b')$ we have $R_2(fab, f'a'b')$. In other words, we need to show that if $r_1 a =_{\beta\eta} a'$ and $r_2 b =_{\beta\eta} b'$ then $r_2(fab) =_{\beta\eta} f'a'b'$. But this follows from the assumption on r_1, r_2.

Hence, $\mathcal{R}^{\xi,\omega}_{\beta\to\texttt{List}(\alpha)\to\beta}(\texttt{fold}\,\tau\,\sigma\,f, \texttt{fold}\,\tau'\,\sigma'\,f')$. Since $R_2(u, r_2 u)$, also

$$\mathcal{R}^{\xi,\omega}_{\texttt{List}(\alpha)\to\beta}(\texttt{fold}\,\tau\,\sigma\,f\,u, \texttt{fold}\,\tau'\,\sigma'\,f'\,(r_2 u)).$$

By Lemma 5.15 we have $l =_{\beta\eta} [a_1, \ldots, a_n]$. We have $R_1(a_i, r_1 a_i)$ for $i = 1, \ldots, n$. Hence $\mathcal{R}^{\xi,\omega}_{\texttt{List}(\alpha)}([a_1, \ldots, a_n], [r_1 a_1, \ldots, r_1 a_n])$. By closure under $\beta\eta$-conversion and Lemma 5.21 we obtain $\mathcal{R}^{\xi,\omega}_{\texttt{List}(\alpha)}(l, \texttt{map}\,\tau\,\tau'\,r_1\,l)$.

Therefore $\mathcal{R}^{\xi,\omega}_{\beta}(\texttt{fold}\,\tau\,\sigma\,f\,u\,l, \texttt{fold}\,\tau'\,\sigma'\,f'\,(r_2 u)\,(\texttt{map}\,\tau\,\tau'\,r_1\,l))$. Hence

$$r_2(\texttt{fold}\,\tau\,\sigma\,f\,u\,l) =_{\beta\eta} \texttt{fold}\,\tau'\,\sigma'\,f'\,(r_2 u)(\texttt{map}\,\tau\,\tau'\,r_1 l).$$ □

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