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Semantic Consistency Proofs for Systems of Illative Combinatory Logic

PhD dissertation

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Author's declaration:
aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Abstract

Illative systems of combinatory logic consist of combinatory logic extended with additional constants intended to represent logical notions. We introduce some strong systems of illative combinatory logic, extending earlier systems of Barendregt, Bunder and Dekkers. This continues Curry's and Bunder's lines of research on illative combinatory logic. We define semantics for illative systems and show our systems consistent by model constructions. We also investigate properties of translations of traditional systems of logic into the corresponding systems of illative combinatory logic. Some of the systems shown consistent in the present work are much stronger than the systems shown consistent by Barendregt, Bunder and Dekkers. In particular, the strongest of our systems essentially incorporates full extensional classical higher-order logic extended with dependent function types, dependent sums, subtypes and W-types, which allows to interpret a great deal of mathematics in this system.

Streszczenie

Systemy illatywnej logiki kombinatorycznej rozszerzają beztypowy rachunek kombinatorów o dodatkowe stałe mające na celu reprezentację pojęć logicznych. W pracy wprowadzamy pewne silne systemy illatywnej logiki kombinatorycznej będące rozszerzeniem wcześniejszych systemów Barendregta, Bundera i Dekkersa. Tym samym kontynuujemy kierunek badań Curry'ego i Bundera nad illatywną logiką kombinatoryczną. Definiujemy semantykę dla systemów illatywnych i poprzez konstrukcje modeli pokazujemy niesprzeczność naszych systemów. Niektóre spośród systemów których niesprzeczność wykazaliśmy są znacznie silniejsze niż systemy Barendregta, Bundera i Dekkersa. W szczególności, najsilniejszy z naszych systemów zawiera pełną ekstensjonalną klasyczną logikę wyższego rzędu rozszerzoną o zależne typy funkcyjne, sumy zależne, podtypy i W-typy.

Słowa kluczowe

rachunek lambda, illatywna logika kombinatoryczna, semantyka

Keywords

lambda calculus, illative combinatory logic, semantics

ACM Computing Classification

F. Theory of Computation
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F.4.1 Mathematical Logic
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Contents

Chapter 1

Introduction

Illative systems of combinatory logic or lambda-calculus consist of type-free combinatory logic or lambda-calculus extended with additional constants intended to represent logical notions. In fact, early systems of combinatory logic and lambda calculus (by Schönfinkel [Sch24], Curry [Cur30] and Church [Chu32, Chu33]) were meant as very simple foundations for logic and mathematics. However, the Kleene-Rosser and Curry paradoxes caused most logicians to abandon this work.

It has proven surprisingly difficult to formulate and show consistent illative systems strong enough to interpret traditional logic. This was accomplished in [BBD93], [DBB98a] and [DBB98b], where several systems were shown complete for the universal-implicational fragment of first-order intuitionistic predicate logic. In [Cza13b] an extension of one system from [BBD93, DBB98a, DBB98b] in which full higher-order classical logic may be interpreted was shown consistent by semantic methods.

The difficulty in proving consistency of illative systems in essence stems from the fact that, lacking a type regime, arbitrary recursive definitions involving logical operators may be formulated, including negative ones. In early systems containing an unrestricted implication introduction rule this was the reason for the Curry's paradox (see Section 3.1). Formulating appropriate and not too cumbersome restrictions is not easy if the fundamental property of allowing unrestricted recursion is to be retained, to which the Bunder (Section 3.2) and Kleene-Rosser (Section 3.3) paradoxes testify.

The fact that in illative systems unrestricted recursion is directly incorporated into the logic is one of the properties that make these systems interesting from the point of view of computer science. In [Cza13c, Cza13d] it is suggested that using illative-like systems may be a viable approach to the problem of handling unrestricted recursion in interactive theorem provers. An advantage of illative systems is that no justifications are needed for formulating unrestricted recursive definitions. One may just introduce a possibly non-well-founded recursive function definition and start reasoning about it within the logic. There is obviously a trade-off – some inference rules need to be restricted by adding premises which essentially state that some terms are "propositions". To be able to derive that some terms are propositions, illative systems include certain "typing rules", i.e., rules for reasoning about which types (categories) a term belongs to. In contrast to traditional systems, however, these rules are internal to the system. The functions do not need to be "typed" a priori, but reasoning about "types" may be interleaved with other reasoning. For instance, one may show typability by induction. This may possibly be an interesting way of reasoning about potentially non-well-founded function definitions in an interactive theorem prover.

The initial motivation of Curry for studying illative combinatory logic was to develop extremely simple foundations for mathematics and logic, which assume as primitive the notions of self-applicable function-in-intension (operation), and stress the very mechanism of definition/combination of concepts. In this approach to the foundations of mathematics, the notion of function takes priority over the notion of set. A set is a special function, whose application to an argument may sometimes be a proposition. The members of a set are those arguments for which the application is a true assertion.

It is important to note, however, that Curry's aim was not merely to provide an alternative foundational system for mathematics, which would compete with the theory of types, set theory, etc. In Curry's view, combinatory logic concerns itself with the ultimate foundations. Its purpose is the analysis of certain notions of such a basic character that they are taken for granted in most other systems of logic. These are, above all, the analysis of the process of substitution, and also the classification of objects into types or categories. Such notions constitute what Curry calls a prelogic. Although very basic and generally presupposed, these notions are not simple and thus they merit further investigation. Moreover, an analysis of the prelogic may shed some light on the sources of paradoxes, and this was also one of Curry's original motivations. See [CFC58, p. 1] and [Cur80, Sel80, Des04].

In systems of illative combinatory logic, there is a priori only a single sort of terms, only a single binary application operation to form composite terms, and only a single form of judgements. The rules of these systems are to have a simple character, without involving complex notions like substitution. The process of substitution, and the categorisations of terms, are performed entirely inside the system.

In this work we develop semantics for various systems of illative combinatory logic and lambda-calculus which are extensions of some systems from [BBD93, DBB98a, DBB98b, Cza13b]. The systems are then shown consistent by constructing models. We also consider natural embeddings of traditional logical systems into corresponding illative systems. Using semantic methods, we investigate soundness and completeness of these translations.

Some of the systems shown consistent in the present work are much stronger than the systems of [BBD93, DBB98a, DBB98b]. In particular, the system $e\mathcal{I}\mathrm{K}\omega$ from Chapter 6 essentially incorporates full extensional classical higher-order logic. The system $\mathcal{I}^+$ from Chapter 7 extends $e\mathcal{I}\mathrm{K}\omega$ by dependent function types, dependent sums, subtypes and W-types.

The system $\mathcal{I}^+$ is rich enough to interpret much of mathematics. Many common type-theoretic constructions are possible. Using dependent sums one may define finite products and (non-dependent) disjoint sums. Using W-types, which originate from Martin-Löf's type theory [ML84], [NPS90, Chapter 15], one may define inductive types, including the type of natural numbers. The derived induction principles associated with inductive types are unrestricted, i.e., it is possible to apply inductive reasoning to terms whose types have not yet been established, thus for instance enabling reasoning about types of terms by induction.

In most previous work the approach is syntactic – consistency is shown by cut-elimination or by analysis of possible forms of derivable terms using grammars. Establishing cut-elimination is more informative than only constructing a model, but for illative systems it also seems much harder. Our methods are semantic. The consistency proofs are not constructive and need much of the power of set theory. In fact, the model construction for the strongest system $\mathcal{I}^+$ assumes the existence of a strongly inaccessible cardinal, so it is not formalisable in ZFC.

The rest of this chapter is organised as follows. In Section 1.1 we provide some background on illative combinatory logic. In Section 1.2 we briefly outline the results obtained in this work. In Section 1.3 we survey previous work related to illative combinatory logic. In Section 1.4 we give an overview of the systems and results from [BBD93, DBB98a, DBB98b].

1.1 Illative combinatory logic

All illative systems we consider (except $\mathcal{I}^+$) come in three variants differing in the underlying reduction system, which is either combinatory logic with weak reduction, (untyped) lambda-calculus with β-reduction or lambda-calculus with $\beta\eta$-reduction (see Section 2.3), with constants from a fixed signature Σ. Since most of the proofs and definitions are the same or very similar for each of the variants, we usually give only a single generic proof or definition, and possibly note the differences for each variant. We use $\mathbb{T}$ to generically denote the set of terms of an illative system, which is either the set of terms of combinatory logic with extra constants from Σ ($\mathbb{T}_{\mathrm{CL}}(\Sigma)$) or the set of terms of lambda-calculus with constants from Σ ($\mathbb{T}_\lambda(\Sigma)$). Analogously, we use $=$ to generically denote $=_w$, $=_\beta$ or $=_{\beta\eta}$, as appropriate. By $\equiv$ we denote syntactic identity of terms (up to α-conversion in lambda-calculus). We use S and K to generically denote either the constants of combinatory logic, or the terms $\lambda xyz.xz(yz)$ and $\lambda xy.x$ in lambda-calculus. We define $\mathsf{I} \equiv \lambda x.x$ in lambda-calculus, or $\mathsf{I} \equiv \mathsf{SKK}$ in combinatory logic. The notation $\lambda x.M$ is used to denote either combinatory abstraction in CL, or abstraction in lambda-calculus.

Illative systems extend combinatory logic (or lambda-calculus) with *illative primitives* representing logical notions. Unlike in most traditional systems of logic, there is no a priori distinction between various categories: propositions (formulas), individual terms, functions, relations, etc. Instead, there are inference rules which allow some categorisations to be performed *inside* the system. Certain illative primitives represent primitive types[1] (categories), and there are combinators which allow the formation of new types. If a term T represents a type, then TX is an assertion that X has type T. In fact, any term may be potentially asserted as a proposition (which does not mean that all terms represent well-formed propositions), and equal terms (in the sense of weak, β-, or $\beta\eta$-equality, as appropriate) are always interchangeable. Intuitively, types represent permissible quantifier ranges – quantification is allowed only over objects of a fixed type. Predicates on a type T, or subsets of T, are represented by functions from T to the type of propositions H.

[1]The notion of "type" is used informally in this section, interchangeably with "category".

The illative primitives need not be constants – they may be composite terms. An illative primitive which is a constant is called an *illative constant*. Below we list some common illative primitives together with an informal explanation of their meaning (cf. [CHS72, §12B2]). Any given illative system may contain any number of these primitives, and possibly some more. All primitives listed here have appeared in previous work on illative combinatory logic. In Chapter 6 and in Chapter 7 we use some additional primitives, which to our knowledge have not been considered before. In what follows, by $X, Y, Z, \ldots$ we denote arbitrary terms from $\mathbb{T}$.

- P Implication. Instead of $\mathsf{P}XY$ we often write $X \supset Y$. Implication is sometimes defined by $\mathsf{P} \equiv \lambda xy.\Xi(\mathsf{K}x)(\mathsf{K}y)$ (see below for an explanation of Ξ).
- $\mathsf{\Lambda}$ Conjunction. Instead of $\mathsf{\Lambda}XY$ we often write $X \wedge Y$.
- V Disjunction. Instead of $\mathsf{V}XY$ we often write $X \vee Y$.
- $\bot$ False proposition.
- $\top$ True proposition. Often defined by $\top \equiv \mathsf{P}\bot\bot$.
- $\neg$ Negation. Often defined by $\neg \equiv \lambda x.\mathsf{P}x\bot$.
- Ξ Restricted generality – a restricted universal quantifier. The term ΞXY is intuitively interpreted as "$X \subseteq Y$", or "for every object Z such that XZ we have YZ", or "for every object Z of type X we have YZ". The notation $\forall x : X \,.\, Y$ is often used to denote $\Xi X(\lambda x.Y)$. Note that x is not bound in X.
- X Restricted existential quantifier. The term $\mathsf{X}YZ$ is intuitively interpreted as "there is an object X such that YX and ZX", or "there exists an object X of type Y such that ZX". The notation $\exists x : Y \,.\, Z$ is often used to denote $\mathsf{X}Y\lambda x.Z$. Note that x is not bound in Y.
- F Functionality (cf. [CFC58, §8C]). The term $\mathsf{F}XYF$ is intuitively interpreted as "F is a function from X to Y", or "for every object Z of type X we have $Y(FZ)$". Functionality is often defined by $\mathsf{F} \equiv \lambda xyf.\Xi x(\lambda z.y(fz))$. Sometimes we write $A \to B$ instead of $\mathsf{F}AB$.
- G Dependent functionality. The term $\mathsf{G}XYF$ is intuitively interpreted as "F is a dependent function which for each Z of type X gives an object of type YZ", or "for every object Z of type X we have $YZ(FZ)$". Dependent functionality is often defined by $\mathsf{G} \equiv \lambda xyf.\Xi x(\lambda z.yz(fz))$.
- F_n Functionality of n arguments. The term $\mathsf{F}_n X_1 \ldots X_n YF$ is intuitively interpreted as "F is an n-argument function from $X_1, \ldots, X_n$ to Y". Usually F_n is defined inductively as follows:

$$\begin{aligned} \mathsf{F}_0 &\equiv \mathsf{I} \\ \mathsf{F}_{n+1} &\equiv \lambda x_1 \ldots x_{n+1} y.\mathsf{F}x_1(\mathsf{F}_n x_2 \ldots x_{n+1} y) \end{aligned}$$

- Q Equality. The term $\mathsf{Q}XY$ is intuitively interpreted as "X and Y are equal".
- H Type of propositions. The term $\mathsf{H}X$ is intuitively interpreted as "X is a proposition". The type of propositions is sometimes defined by $\mathsf{H} \equiv \lambda x.\mathsf{P}xx$ or by $\mathsf{H} \equiv \lambda x.\mathsf{L}(\mathsf{K}x)$.

L Category of types. The term $\mathsf{L}X$ is intuitively interpreted as "X is a type" or "X represents a permissible range of quantification". The category of types is sometimes defined by $\mathsf{L} \equiv \lambda x.\Xi xx$.

A Type of individuals. When interpreting first-order logic this type represents the first-order universe.

E Universal category – the type of all objects. The assertion $\mathsf{E}X$ should be true for any object X.

Using illative primitives, it is possible to interpret ordinary logic in illative combinatory logic. For instance, a first-order sentence

$$\forall x(r(x) \to s(f(x), g(x)) \wedge r(f(x)))$$

is translated as the statement

$$\forall x : \mathsf{A} \,.\, rx \supset s(fx)(gx) \wedge r(fx)$$

which is

$$\Xi\mathsf{A}(\lambda x.\mathsf{P}(rx)(\mathsf{\Lambda}(s(fx)(gx))(r(fx))))$$

where r, s, f, g are constants corresponding to the relation and function symbols from the first-order language, and A represents the first-order universe.

In this work we treat only natural deduction formulations of illative systems. In case of illative combinatory logic, it is not always easy to formulate Hilbert-style or Gentzen-style systems equivalent to a given natural deduction system (the papers [Bun79, BD08] deal with a similar issue). In the present work we do not concern ourselves with this problem. Actually, in view of the results of Section 3.3 it seems plausible that our strongest system $\mathcal{I}^+$ does not have any equivalent Hilbert-style formulation.

In an illative system judgements have the form $\Gamma \vdash X$ where Γ is a finite set of terms and X is a term.[2] If X is a term and Γ a set of terms, then by Γ, X we denote $\Gamma \cup \{X\}$. For an infinite set of terms Γ we write $\Gamma \vdash X$ if there exists a finite subset $\Gamma' \subseteq \Gamma$ with $\Gamma' \vdash X$.

[2] Thus our usage of the symbol $\vdash$ differs somewhat from its usage by Curry. Curry mostly considers (essentially) Hilbert-style systems. The symbol $\vdash$ then denotes a "unary predicate" such that $\vdash X$ for a term X is a meta-level statement meaning "X is provable (in the system under consideration; with no additional hypotheses)". Then $X_1, \ldots, X_n \vdash X$ is only an abbreviation for the meta-level statement "X is provable after adjoining $X_1, \ldots, X_n$ to the list of axioms". Our form of judgements is more complex. Strictly speaking, we need two syntactic categories: one for terms and one for finite sets of terms. Hence our systems are not completely formalised in the sense of [CFC58, §1E5]. Since we are not so much interested in analysing prelogic as in incorporating unrestricted recursion into a system of logic, we shall not concern ourselves too much with such issues. See [CFC58, Chapter 1], [Cur80, §8] and [Cur41b, §2-3] for a more thorough and precise discussion of Curry's conception of formal systems and of the meaning of $\vdash$. In fact, by the results of Section 3.3 it seems plausible that the strongest of our systems incorporating unrestricted induction rules has no reasonable Hilbert-style formulations. We are not concerned by this situation, and in this respect our approach differs from that of Curry. For Curry the ("naturalness" preserving) reduction of certain general types of formal systems to completely formalised (in the sense of [CFC58, §1E5]) systems of preferably strictly finite structure was one of the main tasks of combinatory logic.

All illative systems are required to include the following axiom (Ax) and the rule (Eq) (cf. the definition of $\mathcal{F}_0$ in [CFC58, §8E]). The rule (Eq) essentially incorporates unrestricted recursion into the system.

$$\frac{}{\Gamma, X \vdash X}\ (\mathrm{Ax}) \qquad \frac{\Gamma \vdash X \quad X = Y}{\Gamma \vdash Y}\ (\mathrm{Eq})$$

Here $X = Y$ is a meta-level side condition. Recall that $=$ denotes either weak, β-, or $\beta\eta$-equality, as appropriate.

Note that, strictly speaking, the rule (Eq) does not have a simple character, because the meta-level side condition is of a complex nature, and in the case of β- or $\beta\eta$-equality in the lambda-calculus it involves the notion of substitution. However, at least in the case of weak equality in combinatory logic, this rule could be broken up into several rules of a simple character, at the cost of introducing the illative primitive Q for equality, or a new form of judgement $\vdash X = Y$. Since our interest lies more in the fact that illative systems incorporate unrestricted recursion directly into the logic, rather than with the aim of analysing prelogic, we shall not concern ourselves too much with such issues.

If an illative system includes one of the illative primitives P, Ξ, F, G, then we require that it incorporates the corresponding elimination rules (either directly or as derived rules).

$$\frac{\Gamma \vdash X \supset Y \quad \Gamma \vdash X}{\Gamma \vdash Y}\ (\mathsf{PE}) \qquad \frac{\Gamma \vdash \Xi XY \quad \Gamma \vdash XZ}{\Gamma \vdash YZ}\ (\Xi\mathrm{E})$$

$$\frac{\Gamma \vdash \mathsf{F}XYF \quad \Gamma \vdash XZ}{\Gamma \vdash Y(FZ)}\ (\mathsf{FE}) \qquad \frac{\Gamma \vdash \mathsf{G}XYF \quad \Gamma \vdash XZ}{\Gamma \vdash YZ(FZ)}\ (\mathsf{GE})$$

It is less clear how introduction rules should look like. Curry's paradox implies that adding the following natural candidate for an introduction rule for P yields an inconsistent system (see Section 3.1).

$$\frac{\Gamma, X \vdash Y}{\Gamma \vdash X \supset Y}\ (\mathrm{DED})$$

Intuitively, the problem is that, a priori, we do not know whether X is a proposition, so $X \supset Y$ may not make any sense. If $X = (X \supset \bot)$ then using the above rule we can derive a contradiction.

A way out of the paradox is to add the illative primitive H, appropriately restrict introduction rules, and add rules to reason about which terms represent propositions. Of course, we would like the restrictions in introduction rules to be as unobtrusive as possible. It would not be difficult to formulate and show consistent an "illative" system in which the restrictions would be so strong as to make it indistiguishable in practice from a system in which terms are a priori assigned to definite syntactic categories (or typed statically), but the point of introducing such a system is dubious. The illative systems we will be concerned with have minimal restrictions in introduction rules – in the sense that removing any of these restrictions yields an inconsistent system.

With regard to illative systems, there are two common notions of consistency [Bun77]: weak and strong consistency. Weak consistency means that there exists a term which is not derivable. Strong consistency means that there exists a term X which is not derivable, and it is provable that X is a proposition, i.e., $\nvdash X$ and $\vdash \mathsf{H}X$. When referring to consistency we shall always mean strong consistency. In fact, for systems introduced in the present work these notions are equivalent.

The introduction rule for P which we shall adopt is the following.

$$\frac{\Gamma, X \vdash Y \quad \Gamma \vdash \mathsf{H}X}{\Gamma \vdash X \supset Y}\ (\mathsf{PI})$$

A visible disadvantage of this rule is that to use it an additional premise $\Gamma \vdash \mathsf{H}X$ needs to be shown, but if we want the rule (Eq) we cannot do much better. However, the "typing rules" for H, i.e., rules for reasoning about which terms represent propositions, will be of such a character that in most cases deriving this premise will be straightforward. This is made more precise in the succeeding chapters. In particular, the soundness of translations of traditional systems of logic into illative combinatory logic shows that additional premises in introduction rules hold as long as we deal only with terms which are translations of terms or formulas of a traditional system. Explicitly deriving the additional premises may be needed only when dealing with terms which do not have direct counterparts in traditional systems.

An advantage of illative systems is that their "typing rules", i.e., rules for reasoning about which types (categories) a term belongs to, are similar to rules in traditional type systems. In fact, these rules are usually generalisations of traditional typing rules. Therefore, in a machine implementation of illative logic, it may be possible to adapt standard type checking or type inference algorithms to obtain algorithms which, in common cases, automatically produce a derivation establishing which type a given term belongs to, and thus dispose of the additional premises in introduction rules. See also [Cza13c, Cza13d].

1.2 Contribution

The main result of this work is that all the illative systems $\mathcal{I}$Jp, $\mathcal{I}$Kp, $\mathcal{I}$J, $\mathcal{I}$K, $\mathcal{I}\mathrm{K}\omega$, $e\mathcal{I}\mathrm{K}\omega$ and $\mathcal{I}^+$ introduced in the following chapters are consistent, i.e., $\bot$ is not derivable in the empty context. To prove this result, for each system we introduce a semantics with respect to which the system is shown to be sound, and then we construct a model. Some of the systems are also shown complete w.r.t. the corresponding semantics. The model constructions are parameterised by corresponding models of traditional systems of logic. They essentially show truth-preserving transformations of models of traditional systems into corresponding models of illative systems. We later use the constructions to show completeness of some translations of traditional systems into illative systems. Soundness of these translations is also show, usually by semantic means. Soundness means that if a judgement of a traditional system is provable, then so is its translation. Completeness means that if the translation of a judgement is provable then so is the original judgement. Below we give a more detailed overview of the contents of the present work and of the obtained results.

In Chapter 2 we provide the necessary background and introduce various notions needed in the subsequent chapters. We also introduce definitions of a few non-standard notions and some simple lemmas concerning these notions. In particular, we define Extended Abstract Reduction Systems, and the notions of coherence and invariance, which are crucial in the model constructions.

In Chapter 3 we present three paradoxes in systems of illative combinatory logic: Curry's paradox, Bunder's paradox, and the Kleene-Rosser paradox. These paradoxes show certain limitations on the rules an illative system may contain. Our treatment of the Kleene-Rosser paradox, though based on earlier work, is new: it reveals an essential incompatibility between an unrestricted induction principle for natural numbers and a Hilbert-style formulation of an illative system.

In Chapter 4 we study two illative systems: the system $\mathcal{I}$Jp of propositional intuitionistic logic, and the system $\mathcal{I}$Kp of propositional classical logic. We develop semantics for both of these system. The models for $\mathcal{I}$Jp are essentially combinatory algebras combined with Kripke frames. The models for $\mathcal{I}$Kp are combinatory algebras with two sets $\mathcal{T}$ and $\mathcal{F}$ of true and false elements of the algebra, with some natural conditions imposed on $\mathcal{T}$ and $\mathcal{F}$. We show that $\mathcal{I}$Jp and $\mathcal{I}$Kp are sound and complete w.r.t the corresponding semantics. We prove the consistency of $\mathcal{I}$Jp and $\mathcal{I}$Kp by constructing models. The model constructions are parameterised by corresponding models for traditional systems. We use the constructions to show and completeness of natural translations of the traditional system NJp of propositional intuitionistic logic into $\mathcal{I}$Jp, and of the traditional system NKp of propositional classical logic into $\mathcal{I}$Kp. Soundness of these translations is also established by semantic arguments.

In Chapter 5 we investigate the intuitionistic first-order illative system $\mathcal{I}$J, and the classical first-order illative system $\mathcal{I}$K. We develop Kripke-style semantics for $\mathcal{I}$J, which extends the semantics for $\mathcal{I}$Jp. We prove that $\mathcal{I}$J is sound and complete w.r.t. this semantics. For $\mathcal{I}$K the natural semantics extending the semantics for $\mathcal{I}$Kp is show to be sound, but we do not know whether it is complete. The problem is that in classical illative systems with quantifiers we have excluded middle only for terms which may be proved to be propositions. This makes it impossible to easily adapt the standard Henkin-style completeness proof. We show that $\mathcal{I}$K is complete w.r.t. a somewhat less natural semantics which allows more than one state. We prove consistency of $\mathcal{I}$J and $\mathcal{I}$K by model constructions, which are parameterised by corresponding models of traditional systems. Like in Chapter 5, the constructions are then used to show completeness of natural translations of traditional intuitionistic first-order logic into $\mathcal{I}$J, and of traditional classical first-order logic into $\mathcal{I}$K. Soundness of the translations is also shown.

In Chapter 6 we study the classical intensional higher-order illative system $\mathcal{I}\mathrm{K}\omega$, and its extensional variant $e\mathcal{I}\mathrm{K}\omega$. We provide natural semantics for both of the systems. We show the systems sound w.r.t. the corresponding semantics. We construct a model for $e\mathcal{I}\mathrm{K}\omega$, which establishes the consistency of both $\mathcal{I}\mathrm{K}\omega$ and $e\mathcal{I}\mathrm{K}\omega$. The construction is parameterised by a standard model for traditional higher-order logic. We show a sound translation from traditional classical intensional (extensional) higher-order logic into $\mathcal{I}\mathrm{K}\omega$ ($e\mathcal{I}\mathrm{K}\omega$). We did not prove the completeness of these translations, because our model construction relies on the fact that the model of traditional higher-order logic by which it is parameterised is a standard model, and traditional higher-order logic is not complete w.r.t. standard semantics.

However, the model construction suffices to derive a limited completeness result: if a translated judgement of higher-order logic is provable in $e\mathcal{I}\mathrm{K}\omega$ then it is valid in all standard models for higher-order logic.

In Chapter 7 we introduce the strongest of our illative systems: the system $\mathcal{I}^+$ which extends $e\mathcal{I}\mathrm{K}\omega$ by a choice operator, universal and empty types, the conditional combinator, subtypes, dependent function types, dependent sums and W-types. The semantics of $\mathcal{I}^+$ is an extension of that for $e\mathcal{I}\mathrm{K}\omega$. The model construction is also an extension of the construction for $e\mathcal{I}\mathrm{K}\omega$. For $\mathcal{I}^+$ we carry out the model construction under the assumption of the existence of a strongly inaccessible cardinal.

1.3 Related work

The subject of combinatory logic began with Schönfinkel's [Sch24], where it is shown how to eliminate bound variables in logical expressions by reducing them to applicative terms built up from the combinators S and K (see Section 2.3.2) and the *Unverträglichkeitsfunktion* U defined by the equation

$$\mathsf{U}fg = \forall x(\neg(fx) \vee \neg(gx)).$$

Actually, in [Sch24] no logical axioms for U are formulated, and no formal system in modern sense is given. The function U is only defined informally by the above equation.

Later Curry formulated systems of logic based on untyped combinatory logic [Cur30, Cur31, Cur32, Cur33, Cur34a, Cur34b], and Church introduced systems of logic based on the untyped lambda-calculus [Chu32, Chu33]. These systems were shown inconsistent by Kleene and Rosser [KR35] (see also Section 3.3). A simpler paradox was later found by Curry [Cur42b] (see Section 3.1).

Curry and his school then started the program of defining systems of illative combinatory logic of varying strength, hoping to ultimately obtain consistent systems strong enough to interpret traditional logic. See [Cur42c] and [CFC58, §8]. Bunder [Bun69, Bun73a, Bun74a, Bun83] introduced restrictions in the rules for illative primitives so that traditional logic may be interpreted in the resulting systems, but their consistency remains open. Some variations on several systems of Bunder were shown consistent in [BBD93, DBB98a, DBB98b, Cza13b].

The monograph [CFC58] contains an introduction to illative combinatory logic, which is followed by a more extensive exposition in the second volume [CHS72]. However, the main system $\mathcal{F}_{21}^*$ of [CHS72] was shown inconsistent in [Bun76]. Later in [BM78, Shu78] this inconsistency result was extended to a larger class of systems similar to $\mathcal{F}_{21}^*$. See Section 3.2.

It is also worthwhile to mention the system $\mathcal{F}_{22}$ introduced by Curry in [Cur73]. In [Cur73] Curry proved this system consistent in a weak sense (every term which occurs in a proof belongs to a class of terms intended to represent propositions). Later Seldin obtained stronger consistency results (normalisation) [Sel75, Sel77a, Sel77b]. The system $\mathcal{F}_{22}$ is essentially a type-free intuitionistic predicate calculus without conjunction, alternation and negation but with quantification over propositional functions. It may be extended to include the remaining connectives and quantifiers [Sel77b]. However, in $\mathcal{F}_{22}$ the illative primitive L (see Section 1.1) is defined by $\mathsf{L} \equiv \mathsf{FEH}$, i.e., the "types" are identified with propositional functions defined on *arbitrary* objects. This makes it impossible to extend $\mathcal{F}_{22}$ to a system allowing quantification over propositions, because of Bunder's paradox ($\vdash \mathsf{LH}$ would imply $\vdash \mathsf{H}(\mathsf{H}X)$ for an arbitrary X; see Section 3.2).

In [Sel00] Seldin proves consistent a system of illative combinatory logic with quantifiers and all propositional connectives except for implication and negation. The rules for the connectives are unrestricted, and the consistency proof is strictly finitary. This gives some evidence to the claim that these are the rules for implication which influence the strength of an illative system. In [Sel00] Seldin also shows consistent an illative system with a restricted set of rules for implication, related to BCK-logic.

A readable introduction to illative combinatory logic, as well as a historical overview, may be found in [Sel09]. Chapter 17 of the book [HS86] also treats illative combinatory logic in an introductory way. The annotated bibliography [Bet99] has an extensive (but by no means complete) section on illative combinatory logic. Also the monographs [CFC58, CHS72] and the articles [BBD93] and [CH09, §5.4] contain additional references and historical remarks. For some philosophical issues concerning illative combinatory logic and a description of Curry's initial motivations see [Cur80, Sel80, Des04] and [CFC58, Chapter 1].

The systems studied in the present work are extensions of some systems from [BBD93, DBB98a, DBB98b, Cza13b], which in turn are based on the work of Bunder [Bun69, Bun73a, Bun74a, Bun83]. The idea of using the illative primitive H to represent the category of propositions dates back to Curry's [Cur42c].

There are many illative systems which differ substantially from those originating in Curry's and Bunder's lines of research. One example are Fitch's systems [Fit74, Fit80a, Fit80b, Fit81], in particular his system Q [Fit74, Fit81]. System Q is strong enough to interpret traditional logic and it was shown consistent in [Fit81], after minor modifications. Essentially, implication introduction is restricted by requiring that the law of excluded middle holds for the antecedent. There are some differences in handling equality, with some restrictions on certain subproofs. Quantification ranges over all terms, and there is a constant N representing the class of natural numbers. There is an axiom to the effect that $\mathsf{N}X$ satisfies the law of excluded middle for arbitrary X, which essentially enables quantification over natural numbers. However, this axiom is incompatible with the conditional combinator (see Section 7.1).

There are also many systems which are not systems of illative combinatory logic, but to some extent incorporate recursion and the notion of self-application without type restrictions. For instance, Feferman's systems of explicit mathematics [Fef75, Fef79, Bee85], in their formulation from [Bee85, Chapter X], are based on Beeson's Logic of Partial Terms [Fef95, Bee85, Chapter VI] and include the axioms of a partial combinatory algebra. One difference from illative systems is that application is partial – illative combinatory logic is based on ordinary total combinatory logic. Another and perhaps even more fundamental difference is that in Feferman's systems there is an a priori syntactic distinction between formulas and terms. In illative systems there is just one syntactic category, and all reasoning about which terms represent propositions is carried out within the system. This property of illative systems makes it hard to construct models.

Another development in partial logics are Farmer's papers [Far90, Far93, FGT93] which introduce higher-order logics with partial functions. However, these papers deal mostly with handling partial functions in higher-order logic, not with general unrestricted recursion. In fact, the language of these theories is typed – they are essentially variants of Church's simple type theory [Chu40].

Aczel's classic [Acz80] introduces Frege structures, giving a semantic account of Frege's logical notion of set, i.e., sets understood as extensions of propositional functions. Frege structures are models of lambda-calculus together with a collection of "propositions" and its subcollection of "truths". Thus, unrestricted recursion and self-application are allowed, and recursion may involve logical operators. In fact, Frege structures may be used to give an interpretation of some first-order illative systems [HS86, Chapter 17]. Aczel's construction of Frege structures is very similar to the simplest of our model constructions for $\mathcal{I}$Kp and $\mathcal{I}$K. The general idea of this construction – a monotone inductive definition of a "truth predicate" – has appeared in many other works, e.g. [Sco75, Kri75, Fit81].

In [Cza11] a semantic treatment of a combination of classical first-order logic with type-free combinatory logic was given. The system of [Cza11] is more complex than Aczel's [Acz80] or than simple illative systems in that it contains the conditional combinator (see Section 7.1), which makes equality dependent on truth values of terms, and the model construction becomes more difficult. Nonetheless, the idea of the construction in [Cza11] is also a monotone inductive definition, but of a term rewriting system. The model construction method from [Cza11] was later significantly revised and extended in [Cza13b, Cza13c, Cza13d] and in the present work. In fact, the basic method of [Cza11] may be traced back to [JS95], which constructs a model for a certain total applicative theory with a non-constructive μ-operator. Applicative theories form the basis of systems of explicit mathematics [JKS99]. They are usually partial, i.e., based on the Logic of Partial Terms.

Systems of illative combinatory logic are also close to Pure Type Systems (PTS). This connection has been explored in [BD05] where some illative-like systems were proven equivalent to more liberal variants of PTSs from [BD01]. Those illative systems, however, differ somewhat from what is in the literature.

1.4 The systems of Barendregt, Bunder and Dekkers

Since the illative systems studied in the present work are essentially extensions of some systems from the papers [BBD93, DBB98a, DBB98b] by Barendregt, Bunder and Dekkers, we shall now give an overview of the systems and results from these papers.

In [BBD93] four systems $\mathcal{I}\mathsf{P}$, $\mathcal{I}\mathsf{F}$, $\mathcal{I}\Xi$ and $\mathcal{I}\mathsf{G}$ of illative combinatory logic are defined. The set of terms in each of them is the set of all untyped lambda-terms extended with the extra illative constants Ξ and L. Other illative primitives are defined as follows:

$$\begin{array}{rcl}\mathsf{P} & \equiv & \lambda xy.\Xi(\mathsf{K}x)(\mathsf{K}y)\\ \mathsf{F} & \equiv & \lambda xyf.\Xi x(\lambda z.y(fz))\\ \mathsf{G} & \equiv & \lambda xyf.\Xi x(\lambda z.yz(fz))\\ \mathsf{H} & \equiv & \lambda x.\mathsf{L}(\mathsf{K}x)\end{array}$$

where $\mathsf{K} \equiv \lambda xy.x$. An abbreviation $X \supset Y$ is adopted for $\mathsf{P}XY$.

The judgements in all four systems have the form $\Gamma \vdash X$ where X is a term and Γ is a finite set of terms. We often write Γ, X instead of the formally correct $\Gamma \cup \{X\}$.

All four systems have the rules from Figure 1.1. The rules specific to each system are presented in Figures 1.2-1.5.

$$\frac{}{\Gamma, X \vdash X}\ (\mathrm{Ax}) \qquad \frac{\Gamma \vdash X \quad X =_{\beta\eta} Y}{\Gamma \vdash Y}\ (\mathrm{Eq})$$

Figure 1.1: Common rules of $\mathcal{I}\mathsf{P}$, $\mathcal{I}\mathsf{F}$, $\mathcal{I}\Xi$ and $\mathcal{I}\mathsf{G}$.

$$\frac{\Gamma \vdash X \supset Y \quad \Gamma \vdash X}{\Gamma \vdash Y}\ (\mathsf{PE}) \qquad \frac{\Gamma, X \vdash Y \quad \Gamma \vdash \mathsf{H}X}{\Gamma \vdash X \supset Y}\ (\mathsf{PI}) \qquad \frac{\Gamma, X \vdash \mathsf{H}Y \quad \Gamma \vdash \mathsf{H}X}{\Gamma \vdash \mathsf{H}(X \supset Y)}\ (\mathsf{P}\mathrm{H})$$

Figure 1.2: Rules of $\mathcal{I}\mathsf{P}$

$$\frac{\Gamma \vdash \mathsf{F}XYZ \quad \Gamma \vdash XV}{\Gamma \vdash Y(ZV)}\ (\mathsf{FE}) \qquad \frac{\Gamma, Xx \vdash Y(Zx) \quad \Gamma \vdash \mathsf{L}X \quad x \notin \mathrm{FV}(\Gamma, X, Y, Z)}{\Gamma \vdash \mathsf{F}XYZ}\ (\mathsf{FI})$$

$$\frac{\Gamma, Xx \vdash \mathsf{L}Y \quad \Gamma \vdash \mathsf{L}X \quad x \notin \mathrm{FV}(\Gamma, X, Y)}{\Gamma \vdash \mathsf{L}(\mathsf{F}XY)}\ (\mathsf{FL})$$

Figure 1.3: Rules of $\mathcal{I}\mathsf{F}$

The systems $\mathcal{I}\mathsf{P}$ and $\mathcal{I}\mathsf{F}$ can represent propositional minimal logic. The systems $\mathcal{I}\Xi$ and $\mathcal{I}\mathsf{G}$ can represent the universal-implicational fragment of first-order intuitionistic logic. For the systems $\mathcal{I}\mathsf{P}$ and $\mathcal{I}\Xi$ the interpretation is direct, while for $\mathcal{I}\mathsf{F}$ and $\mathcal{I}\mathsf{G}$ it follows the propositions-as-types paradigm by translating derivations to combinators inside the system. In [BBD93] it is shown that the two direct translations are complete, and in [DBB98a, DBB98b] completeness is shown for the indirect translations. This establishes strong consistency of all the systems $\mathcal{I}\mathsf{P}$, $\mathcal{I}\mathsf{F}$, $\mathcal{I}\Xi$ and $\mathcal{I}\mathsf{G}$.

We shall now outline the translation from the universal-implicational fragment of first-order intuitionistic logic into $\mathcal{I}\Xi$, to give a flavour of how such a translation looks like. This translation is very similar to translations used later in the present work.

First, we define the system PRED of universal-implicational first-order intuitionistic logic. Let Σ be a first-order signature, and V a set of variables. The set of terms of PRED, denoted $\mathbb{T}$, is defined inductively:

- $V \subseteq \mathbb{T}$,
- if $f \in \Sigma$ is an n-ary function symbol (possibly $n = 0$) and $t_1, \ldots, t_n \in \mathbb{T}$, then $f(t_1, \ldots, t_n) \in \mathbb{T}$.

$$\frac{\Gamma \vdash \Xi XY \quad \Gamma \vdash XV}{\Gamma \vdash YV} \ (\Xi\text{E}) \qquad \frac{\Gamma, Xx \vdash Yx \quad \Gamma \vdash \mathsf{L}X \quad x \notin \text{FV}(\Gamma, X, Y)}{\Gamma \vdash \Xi XY} \ (\Xi\text{I})$$

$$\frac{\Gamma, Xx \vdash \mathsf{H}(Yx) \quad \Gamma \vdash \mathsf{L}X \quad x \notin \text{FV}(\Gamma, X, Y)}{\Gamma \vdash \mathsf{H}(\Xi XY)} \ (\Xi\text{H})$$

Figure 1.4: Rules of $\mathcal{I}\Xi$

$$\frac{\Gamma \vdash \mathsf{G}XYZ \quad \Gamma \vdash XV}{\Gamma \vdash YV(ZV)} \ (\mathsf{GE}) \qquad \frac{\Gamma, Xx \vdash Yx(Zx) \quad \Gamma \vdash \mathsf{L}X \quad x \notin \text{FV}(\Gamma, X, Y, Z)}{\Gamma \vdash \mathsf{G}XYZ} \ (\mathsf{GI})$$

$$\frac{\Gamma, Xx \vdash \mathsf{L}(Yx) \quad \Gamma \vdash \mathsf{L}X \quad x \notin \text{FV}(\Gamma, X, Y)}{\Gamma \vdash \mathsf{L}(\mathsf{G}XY)} \ (\mathsf{GL})$$

Figure 1.5: Rules of $\mathcal{I}\mathsf{G}$

The set of formulas of PRED, denoted $\mathbb{F}$, is defined inductively:

- if $r \in \Sigma$ is an n-ary relation symbol and $t_1, \ldots, t_n \in \mathbb{T}$, then $r(t_1, \ldots, t_n) \in \mathbb{F}$,
- if $\varphi, \psi \in \mathbb{F}$ then $\varphi \supset \psi \in \mathbb{F}$,
- if $x \in V$ and $\varphi \in \mathbb{F}$ then $\forall x \varphi \in \mathbb{F}$.

The judgements of PRED have the form $\Delta \vdash \varphi$, where Δ is a finite set of formulas and φ is a formula. We use the notation Δ, φ for $\Delta \cup \{\varphi\}$. The rules of PRED are presented in Figure 1.6.

$$\frac{}{\Delta, \varphi \vdash \varphi} \ (\text{Ax})$$

$$\frac{\Delta \vdash \varphi \supset \psi \quad \Delta \vdash \varphi}{\Delta \vdash \psi} \ (\supset_e) \qquad \frac{\Delta, \varphi \vdash \psi}{\Delta \vdash \varphi \supset \psi} \ (\supset_i)$$

$$\frac{\Delta \vdash \forall x \varphi \quad t \in \mathbb{T}}{\Delta \vdash \psi[x/t]} \ (\forall_e) \qquad \frac{\Delta \vdash \varphi \quad x \notin \text{FV}(\Delta)}{\Delta \vdash \forall x \varphi} \ (\forall_i)$$

Figure 1.6: Rules of PRED

For the translation, we assume that the set of terms of $\mathcal{I}\Xi$ contains each element of Σ as a constant, each variable from V as a variable, and there is a constant A representing the first-order universe. The translation $\lceil - \rceil$ from the terms and formulas of PRED into the set of terms of $\mathcal{I}\Xi$ is defined inductively as follows:

- $\lceil x \rceil = x$,

- $\lceil f(t_1, \ldots, t_n) \rceil = f \lceil t_1 \rceil \ldots \lceil t_n \rceil$,
- $\lceil r(t_1, \ldots, t_n) \rceil = r \lceil t_1 \rceil \ldots \lceil t_n \rceil$,
- $\lceil \varphi \supset \psi \rceil = \lceil \varphi \rceil \supset \lceil \psi \rceil$,
- $\lceil \forall x \varphi \rceil = \Xi \mathsf{A}(\lambda x. \lceil \varphi \rceil)$.

The function $\lceil - \rceil$ is extended to sets of formulas by defining $\lceil \Delta \rceil = \{\lceil \varphi \rceil \mid \varphi \in \Delta\}$. We define the context-providing mapping Γ from sets of formulas to sets of terms of $\mathcal{I}\Xi$ as follows, where F_n is defined as in Section 1.1:

- $\mathsf{F}_n \mathsf{A} \ldots \mathsf{A} \mathsf{A} f \in \Gamma(\Delta)$ for $f \in \Sigma$ an n-ary function symbol,
- $\mathsf{F}_n \mathsf{A} \ldots \mathsf{A} \mathsf{H} r \in \Gamma(\Delta)$ for $r \in \Sigma$ an n-ary relation symbol,
- $\mathsf{A} x \in \Gamma(\Delta)$ for every $x \in \mathrm{FV}(\Delta)$,
- $\mathsf{A} y \in \Gamma(\Delta)$ for some fresh $y \notin \mathrm{FV}(\Delta)$.

The soundness and completeness of the translation are stated in the following theorem. Soundness is the implication from left to right, and completeness is the implication in the other direction.

Theorem 1.4.1 ([BBD93]). $\Delta \vdash_{\mathrm{PRED}} \varphi$ *iff* $\Gamma(\Delta, \varphi), \lceil \Delta \rceil \vdash_{\mathcal{I}\Xi} \lceil \varphi \rceil$.

The methods of [BBD93, DBB98a, DBB98b] are purely syntactic. Soundness of the translations is not difficult to establish, and it is shown by a relatively straightforward induction on the length of derivations. The more difficult completeness, which also implies consistency, is shown by analysing possible forms of derivable terms using grammars.

In contrast, the approach of the present work is semantic. By constructing models, we show consistency of some strong illative systems which are extensions of $\mathcal{I}\mathsf{P}$, $\mathcal{I}\mathsf{F}$, $\mathcal{I}\Xi$ and $\mathcal{I}\mathsf{G}$, with minor modifications. We also show soundness and completeness of direct translations of traditional systems of logic into some of our illative systems.

Chapter 2

Preliminaries

In this chapter we provide the necessary background and introduce various notions needed in the subsequent chapters. The first section is devoted to fixpoint definitions, including two variants of the Knaster-Tarski fixpoint theorem. In the second section we review a few set-theoretic notions needed in Chapter 7, in particular the strongly inaccessible cardinals. In the third section we review basic notions and results in lambda-calculus and combinatory logic. We also introduce definitions of a few non-standard notions and some simple lemmas concerning these notions. In the last section we give a presentation of some traditional systems of logic.

2.1 Fixpoint definitions

Definition 2.1.1. Let A be a set. We define a partial order $\leq_I$ on $\mathbb{P}(A)^I$ coordinatewise: $f \leq_I g$ iff $\forall_{i \in I} f(i) \subseteq g(i)$. The *supremum* $\bigvee X \in \mathbb{P}(A)^I$ of $X \subseteq \mathbb{P}(A)^I$ is defined by: $(\bigvee X)(i) = \bigcup_{f \in X} f(i)$. A function $F : \mathbb{P}(A)^I \to \mathbb{P}(A)^I$ is *monotone* if $f \leq_I g$ implies $F(f) \leq_I F(g)$. A *fixpoint* of F is an $r \in \mathbb{P}(A)^I$ such that $F(r) = r$. The *least fixpoint* of F is a fixpoint r such that $r \leq_I s$ for any other fixpoint s of F.

The following is a special case of the well-known Knaster-Tarski fixpoint theorem [Tar55, Kna28].

Theorem 2.1.2 (Tarski fixpoint theorem)**.** *If $F : \mathbb{P}(A)^I \to \mathbb{P}(A)^I$ is a monotone function then there exists the least fixpoint r of F. Moreover, r may be characterised by the transfinite inductive definition: $F^\alpha = F(F^{<\alpha})$ for all ordinals α, where $F^{<\alpha} = \bigvee_{\beta<\alpha} F^\beta$, and $r = F^\zeta$ for the smallest ordinal ζ such that $F^\zeta = F^{<\zeta}$.*

The above theorem allows us to define sets of relations by mutually recursive conditional rules. For instance, we give a definition of a binary relation $\succ$ on the set of terms $\mathbb{T}$ by the following conditional rules:

- $\top \succ \top$,
- $\bot \succ \bot$,

- $\bigwedge XY \succ \top$ if $X \succ \top$ and $Y \succ \top$,
- $\bigwedge XY \succ \bot$ if $X \succ \bot$ or $Y \succ \bot$.

It is to be understood that $\succ$ is the least fixpoint of the monotone operator

$$F : \mathbb{P}(\mathbb{T} \times \mathbb{T}) \to \mathbb{P}(\mathbb{T} \times \mathbb{T})$$

determined in an obvious way from the rules:

$$\begin{array}{rcl} F(R) & = & \{\langle M, \top\rangle \mid (M \equiv \top) \vee (M \equiv \bigwedge XY \wedge \langle X, \top\rangle \in R \wedge \langle Y, \top\rangle \in R)\} \cup \\ & & \{\langle M, \bot\rangle \mid (M \equiv \bot) \vee (M \equiv \bigwedge XY \wedge (\langle X, \bot\rangle \in R \vee \langle Y, \bot\rangle \in R))\} \end{array}$$

If a set of conditional rules determines an operator which is monotone, then we say that the rules are monotone. If a relation $\succ$ is defined by monotone rules, then we use the notations $\succ^\alpha = F^\alpha$ and $\succ^{<\alpha} = F^{<\alpha}$, where F^α and $F^{<\alpha}$ are as in Theorem 2.1.2. The relation $\succ^\alpha$ is called the *α-th approximant* of $\succ$. The least ordinal ζ such that $\succ^\zeta = \succ$ is called the *closure ordinal* of the definition of $\succ$.

Also the following variant of the Tarski fixpoint theorem will be used in some model constructions.

Theorem 2.1.3. *Let $\{X_s^\alpha\}_{s \in S}$ be a family of subsets of a set A for each ordinal α, i.e., $X_s^\alpha \subseteq A$ for all $s \in S$ and all ordinals α. Let $X_s^{<\alpha} = \bigcup_{\beta<\alpha} X_s^\beta$. If for all $s \in S$ and all $\alpha \leq \beta$ we have $X_s^\alpha \subseteq X_s^\beta$, then there exists an ordinal ζ such that $X_s^\zeta = X_s^{<\zeta}$ for each $s \in S$.*

2.2 Set theory

In this section we review a few set-theoretic notions needed in Chapter 7. We assume familiarity with basic set theory, including cardinal arithmetic. A standard reference for set theory is [Jec02].

Below we use κ, λ for cardinals and α, β for ordinals. By $|x|$ we denote the cardinality of a set x, i.e., the cardinal which is equinumerous with x. Recall that in the Zermelo-Fraenkel set theory with choice (ZFC) a cardinal κ is an ordinal which is not equinumerous with any ordinal $\alpha < \kappa$. Moreover, in ZFC each ordinal α is equal to the set of all ordinals $\beta < \alpha$.

Definition 2.2.1. The *cumulative hierarchy* is an ordinal-indexed sequence of sets V_α defined as follows:

- $V_0 = \emptyset$,
- $V_{\alpha+1} = \mathbb{P}(V_\alpha)$,
- $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ if α is a limit ordinal.

The *rank* of a set x is the least ordinal α such that $x \in V_\alpha$.

The last point in the following lemma implies that the notion of rank is well-defined.

Lemma 2.2.2.

1. *$V_\alpha \subseteq V_\beta$ for $\alpha \leq \beta$.*
2. *If $x \in V_\alpha$ then $x \subseteq V_\alpha$.*
3. *For every set x there is an ordinal α such that $x \in V_\alpha$.*

Definition 2.2.3. A cardinal κ is a *strong limit* if for any cardinal $\lambda < \kappa$ we have $2^\lambda < \kappa$. An infinite cardinal κ is *regular* if there is no $A \subseteq \kappa$ with $\sup A = \kappa$ and $|A| < \kappa$. A cardinal is *strongly inaccessible* if it is uncountable, regular and a strong limit.

In ZFC the existence of strongly inaccessible cardinals cannot be proven (provided ZFC is consistent). In fact, the theory ZFC+SI, which is ZFC plus the axiom "there exists a strongly inaccessible cardinal", proves the consistency of ZFC. For each strongly inaccessible cardinal κ, the set V_κ is a model of ZFC.

Definition 2.2.4. A set U is a *Grothendieck universe* if it satisfies the following:

1. if $x \in U$ then $x \subseteq U$,
2. if $x \in U$ then $\mathbb{P}(x) \in U$,
3. if $x \in U$ then $\{x\} \in U$,
4. if $I \in U$ and $f \in U^I$ then $\bigcup_{i \in I} f(i) \in U$,
5. $\omega \in U$.

The intuition behind a Grothendieck universe is that it is a set U such that all standard operations of set theory (union, power set, etc.) may be performed on its elements with the results still in U. This intuition is validated by the following lemma.

Lemma 2.2.5. *For any Grothendieck universe U the following conditions hold:*

1. *if $x \subseteq y \in U$ then $x \in U$,*
2. *if $x, y \in U$ then $x \cup y \in U$,*
3. *if $x, y \in U$ then $\{x, y\} \in U$,*
4. *if $x, y \in U$ then $\langle x, y \rangle \in U$,*
5. *if $x, y \in U$ then $x \times y \in U$,*
6. *if $x, y \in U$ then $x^y \in U$,*
7. *if $I \in U$ and $f \in U^I$ then $\prod_{i \in I} f(i) \in U$,*
8. *if $I \in U$ and $f \in U^I$ then $f \in U$.*

Proof.

1. Observe that $x \in \mathbb{P}(y) \in U$.
2. Since $2 \in \omega \in U$, we have $2 \in U$. Now observe that $x \cup y = \bigcup_{i \in 2} f(i)$, where $f \in U^2$ is such that $f(0) = x$ and $f(1) = y$.

3. Observe that $\{x, y\} = \{x\} \cup \{y\}$.
4. Observe that $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$, assuming the standard Kuratowski definition of an ordered pair.
5. Observe that $x \times y \subseteq \mathbb{P}(\mathbb{P}(x \cup y))$.
6. Observe that $x^Y \subseteq \mathbb{P}(x \times y)$.
7. Observe that $\prod_{i\in I} f(i) \subseteq \mathbb{P}(I \times \bigcup_{i\in I} f(i))$.
8. Observe that $f \in (\bigcup_{i\in I}\{f(i)\})^I$.

□

The next lemma implies that the existence of a Grothendieck universe is equivalent to the existence of a strongly inaccessible cardinal.

Lemma 2.2.6. *A set U is a Grothendieck universe iff $U = V_\kappa$ for some strongly inaccessible cardinal κ.*

Proof. See [Wil69]. □

2.3 Rewriting, lambda-calculus, and combinatory logic

2.3.1 Abstract rewriting

Definition 2.3.1. An *Extended Abstract Reduction System* (EARS) is a tuple $\langle A, \to, \{\succ_i\}_{i\in I}\rangle$ where A is a *carrier* set, $\to$ a binary *contraction relation* on A, and $\{\succ_i\}_{i\in I}$ is a family of binary *representation relations* with $\succ_i \in \mathbb{P}(A \times B_i)$ for some set B_i, $i \in I$. We write $\to_R$ for the contraction relation of an EARS R. When A is obvious from the context we sometimes say that $\langle \to, \{\succ_i\}_{i\in I}\rangle$ is an EARS. When $I = \{\succ_1, \ldots, \succ_n\}$ we sometimes say that $\langle A, \to, \succ_1, \ldots, \succ_n\rangle$ or $\langle \to, \succ_1, \ldots, \succ_n\rangle$ is an EARS. We also often confuse an EARS with its rewrite relation, particularly when the family of representation relations is empty.

Let $\to$ be a binary relation. By $\xrightarrow{*}$ we denote the transitive-reflexive closure, by $\xrightarrow{\equiv}$ the reflexive closure, and by $\leftarrow$ the inverse of $\to$.

We often write expressions of the form, e.g., $t_1 \to_1 \cdot \to_2 t_2 \to_2 \cdot \to_1 t_3$, which means: there exist s_1, s_2 such that $t_1 \to_1 s_1 \to_2 t_2 \to_2 s_2 \to_1 t_3$. In a statement of the form "$t_1 \leftarrow \cdot \to t_2$ implies $t_1 \to \cdot \leftarrow t_2$" the variables t_1, t_2 are implicitly universally quantified, e.g., the above statement means "for all t_1, t_2, if there exists s such that $t_1 \leftarrow s \to t_2$, then there exists s' such that $t_1 \to s' \leftarrow t_2$". We write $t \leadsto_i s$ if $t \xrightarrow{*} \cdot \succ_i s$.

We say that a binary relation $\to$ has the *diamond property* if $t_1 \leftarrow \cdot \to t_2$ implies $t_1 \to \cdot \leftarrow t_2$. We say that $\to$ is *confluent* if $\xrightarrow{*}$ has the diamond property. We say that $\to_1$ and $\to_2$ have the *commuting diamond property* if $t_1 \leftarrow_1 \cdot \to_2 t_2$ implies $t_1 \to_1 \cdot \leftarrow_2 t_2$. We say that $\to_1$ and $\to_2$ *commute* if $\xrightarrow{*}_1$ and $\xrightarrow{*}_2$ have the commuting diamond property. Our definition of commuting relations differs from [Bar84] but it is consistent with [Ter03, BN99].

We say that $\to$ *preserves* $\succ$ if $t \leftarrow \cdot \succ s$ implies $t \succ s$.

An EARS $\langle A, \to, \{\succ_i\}_{i\in I}\rangle$ is *coherent* if

1. $\to$ is confluent,
2. $\to$ preserves $\succ_i$, for each $i \in I$,
3. each $\succ_i$ is a partial function, i.e., if $t \succ_i s$ and $t \succ_i s'$ then $s \equiv s'$.

Two EARSs $\langle A, \to^1, \{\succ_i^1\}_{i\in I}\rangle$ and $\langle B, \to^2, \{\succ_i^2\}_{i\in I}\rangle$ are *mutually coherent* if

1. $\to^1$ and $\to^2$ commute,
2. $\to^1$ preserves $\succ_i^2$ for $i \in I$,
3. $\to^2$ preserves $\succ_i^1$ for $i \in I$,
4. if $t \succ_i^1 s$ and $t \succ_i^2 s'$ then $s \equiv s'$.

Intuitively, $t \succ_i s$ is interpreted as "t is represented by s in i". Most often, i will be a type. In other words, if $t \succ_i s$ then t treated as an object of type i "behaves" exactly like s.

Lemma 2.3.2. *Let $\to$, $\to_1$, $\to_2$ be binary relations.*

1. *If $\stackrel{\equiv}{\to}_1$ and $\stackrel{\equiv}{\to}_2$ have the commuting diamond property, then $\to_1$ and $\to_2$ commute.*
2. *If $\stackrel{\equiv}{\to}$ has the diamond property, then $\to$ is confluent.*

The following lemma is a generalisation of the well-known Hindley-Rosen lemma (see e.g. [Bar84, Proposition 3.3.5] or [Ter03, Exercise 1.3.4]). The Hindley-Rosen lemma is obtained by taking both families to be $\{\to_1, \to_2\}$.

Lemma 2.3.3 (General Hindley-Rosen lemma)**.** *Let $\{\to_i^1\}_{i\in I}$ and $\{\to_j^2\}_{j\in J}$ be two families of binary relations on a set A. If for all $i \in I$ and all $j \in J$, the relations $\to_i^1$ and $\to_j^2$ commute, then $\bigcup_{i\in I} \to_i^1$ and $\bigcup_{j\in J} \to_j^2$ commute.*

Proof. See Figure 2.1. □

Lemma 2.3.4. *If for any s_1, s_2 the condition $s_1 \leftarrow_1 \cdot \to_2 s_2$ implies $s_1 \stackrel{*}{\to}_2 \cdot \stackrel{\equiv}{\leftarrow}_1 s_2$, then $\to_1$ and $\to_2$ commute.*

Proof. By a simple diagram chase. □

The following two lemmas will often be used implicitly when working with coherent EARSs.

Lemma 2.3.5. *If an EARS $\langle A, \to, \{\succ_i\}_{i\in I}\rangle$ is coherent, then $t \rightsquigarrow_i s$ iff $t \stackrel{*}{\leftrightarrow} \cdot \succ_i s$.*

Proof. The implication from left to right is obvious. For the other direction, suppose that $t \stackrel{*}{\leftrightarrow} t' \succ_i s$. Then by confluence of $\to$ there is r such that $t \stackrel{*}{\to} r$ and $t' \stackrel{*}{\to} r$. Because $\to$ preserves $\succ_i$ we still have $r \succ_i s$. So $t \stackrel{*}{\to} r \succ_i s$, i.e., $t \rightsquigarrow_i s$. □

Lemma 2.3.6. *Suppose an EARS $\langle A, \to, \{\succ_i\}_{i\in I}\rangle$ is coherent. If $t \rightsquigarrow_i s_1$ and $t \rightsquigarrow_i s_2$, then $s_1 \equiv s_2$.*

Proof. Follows directly from definitions. □

Lemma 2.3.7. *An EARS R is coherent iff R and R are mutually coherent.*

Proof. Follows directly from definitions. □

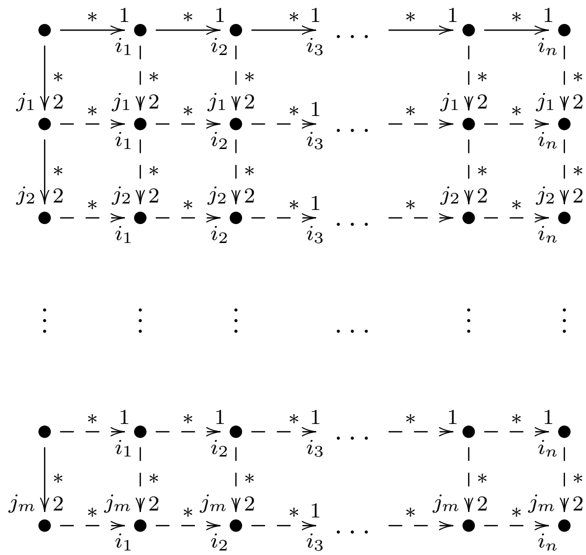

Figure 2.1: General Hindley-Rosen lemma

2.3.2 Lambda-calculus and combinatory logic

We now review some basic definitions and results in lambda-calculus and combinatory logic. A standard reference is [Bar84].

Definition 2.3.8. The set of *lambda-terms* over a set of constants Σ, denoted $\mathbb{T}_\lambda(\Sigma)$, is defined by the grammar

$$\mathbb{T}_\lambda \quad ::= \quad \Sigma \mid V \mid (\lambda V\, \mathbb{T}_\lambda) \mid (\mathbb{T}_\lambda \mathbb{T}_\lambda)$$

where V is a countably infinite set of variables. We write $\mathbb{T}_\lambda$ instead of $\mathbb{T}_\lambda(\Sigma)$ when Σ is clear or irrelevant, and we leave out spurious brackets. By $\mathrm{FV}(X)$ we denote the set of free variables of a term X. We treat lambda-terms up to α-equivalence, i.e., up to renaming of bound variables. By $\equiv$ we denote identity of terms (up to α-equivalence). Substitution $X[x/Y]$ of a term Y for all free occurences of x in X is defined in the expected way, avoiding variable capture. By $X[x_1/Y_1, \ldots, x_n/Y_n]$ we denote simultaneous substitution of $Y_1, \ldots, Y_n$ for $x_1, \ldots, x_n$, avoiding variable capture. The binary relation $\to_\beta$ on $\mathbb{T}_\lambda$ of *β-contraction* is the compatible closure of the β-rule

$$(\lambda x.X)Y \to_\beta X[x/Y].$$

The relation $\stackrel{*}{\to}_\beta$ of *β-reduction* is the transitive-reflexive closure of β-contraction. Analogously, we define *η-contraction* and *η-reduction* using the η-rule

$$\lambda x.Xx \to_\eta X \text{ if } x \notin \mathrm{FV}(X)$$

The relations of *$\beta\eta$-contraction* and *$\beta\eta$-reduction* are defined using both rules. We write $=_\beta$, $=_\eta$, $=_{\beta\eta}$ for the least equivalence relation containing $\to_\beta$, $\to_\eta$, $\to_{\beta\eta}$, respectively.

The set of *combinatory terms* over a set of constants Σ, denoted $\mathbb{T}_{\mathrm{CL}}(\Sigma)$, is defined by the grammar

$$\mathbb{T}_{\mathrm{CL}} \quad ::= \quad \Sigma \mid V \mid \mathsf{K} \mid \mathsf{S} \mid (\mathbb{T}_{\mathrm{CL}}\mathbb{T}_{\mathrm{CL}})$$

where V is a countably infinite set of variables and K, S are constants not present in Σ. We write $\mathbb{T}_{\mathrm{CL}}$ instead of $\mathbb{T}_{\mathrm{CL}}(\Sigma)$ when Σ is clear or irrelevant, and we omit spurious brackets. The notations FV and $\equiv$ are defined as for lambda-terms. The relation $\to_w$ of *weak contraction* is the compatible closure of the rules

$$\begin{array}{rcl} Kxy & \to & x \\ Sxyz & \to & xz(yz) \end{array}$$

The relation $\overset{*}{\to}_w$ of *weak reduction* is the transitive-reflexive closure of weak contraction. By $=_w$ we denote the least equivalence relation containing $\to_w$.

We use the notation $\mathsf{I} \equiv \mathsf{SKK}$. The term I is called the *identity combinator*.

Theorem 2.3.9. *The following conditions hold.*

1. *The relations $\to_\beta$, $\to_{\beta\eta}$ and $\to_w$ are confluent.*
2. *In lambda-calculus, for every term X there exists a term M such that $M =_\beta X[z/M]$. The same holds with $=_w$ in combinatory logic.*

Proof. See [Bar84]. □

The above proposition states two main properties of lambda-calculus and combinatory logic that we will need. The second of these properties essentially implies that these systems enable unrestricted recursive definitions. We will often use the second property implicitly to define terms by recursive equations.

Definition 2.3.10. For a term $X \in \mathbb{T}_{\mathrm{CL}}$ and a variable x, the *combinatory abstraction* $\lambda^* x.X$ is defined inductively:

- $\lambda^* x.x \equiv \mathsf{I}$,
- $\lambda^* x.X \equiv \mathsf{K}X$ if $x \notin \mathrm{FV}(X)$,
- $\lambda^* x.XY \equiv \mathsf{S}(\lambda x^*.X)(\lambda x^*.Y)$.

We define a translation $(-)_{\mathrm{CL}} : \mathbb{T}_\lambda(\Sigma) \to \mathbb{T}_{\mathrm{CL}}(\Sigma)$ inductively:

- $(x)_{\mathrm{CL}} \equiv x$, for $x \in V$,
- $(c)_{\mathrm{CL}} \equiv c$, for $c \in \Sigma$,
- $(XY)_{\mathrm{CL}} \equiv (X)_{\mathrm{CL}}(Y)_{\mathrm{CL}}$,
- $(\lambda x.X)_{\mathrm{CL}} \equiv \lambda x^*.(X)_{\mathrm{CL}}$.

A translation $(-)_\lambda : \mathbb{T}_{\mathrm{CL}} \to \mathbb{T}_\lambda$ is defined inductively:

- $(x)_\lambda \equiv x$, for $x \in V$,
- $(c)_\lambda \equiv c$, for $c \in \Sigma$,

- $(\mathsf{K})_\lambda \equiv \lambda xy.x$,
- $(\mathsf{S})_\lambda \equiv \lambda xyz.xz(yz)$,
- $(XY)_\lambda \equiv (X)_\lambda(Y)_\lambda$.

We often write $\lambda x.X$ instead of $\lambda^* x.X$, when it is clear that $X \in \mathbb{T}_{\mathrm{CL}}$, or to generically denote an abstraction when it is irrelevant whether we work in lambda-calculus or combinatory logic.

Lemma 2.3.11.

1. $(\lambda^* x.X)Y \xrightarrow{*}_w X[x/Y]$.
2. $((X)_{\mathrm{CL}})_\lambda =_\beta X$.

Lemma 2.3.12. *If $x \neq y$ and $x \notin \mathrm{FV}(Y)$ then $\lambda^* x.X[y/Y] \equiv (\lambda^* x.X)[y/Y]$.*

Proof. Induction on the structure of X. □

Lemma 2.3.13. $(Y[x/X])_{\mathrm{CL}} \equiv (Y)_{\mathrm{CL}}[x/(X)_{\mathrm{CL}}]$

Proof. Induction on the structure of Y, using Lemma 2.3.12. □

As remarked in Section 1.1, the illative systems we shall deal with in the following chapters may be based either on lambda-calculus with β-reduction or η-reduction, or on combinatory logic with weak equality. Usually it does not make much difference in definitions or proofs which of these systems is used. This is why we give generic definitions and proofs for any of these systems, or just specialised definitions and proofs for only one of them, and possibly note the differences with the others.

2.3.3 Reduction systems

Definition 2.3.14. A *reduction system* is an EARS whose carrier is either $\mathbb{T}_\lambda(\Sigma)$ or $\mathbb{T}_{\mathrm{CL}}(\Sigma)$, for some Σ. We usually treat a reduction system as a pair $\langle \to, \{\succ_i\}_{i\in I}\rangle$, leaving out the carrier. A reduction system $\langle \to, \{\succ_i\}_{i\in I}\rangle$ is *invariant* when for any $i, j \in I$ such that $\succ_i \in \mathbb{P}(\mathbb{T}\times\mathbb{T})$ the following condition holds:

- if $t \succ_i s$ and $us \rightsquigarrow_j s'$ then $ut \rightsquigarrow_j s'$.

A reduction system $\langle \to, \{\succ_i\}_{i\in I}\rangle$ is *closed under substitution* when the following conditions hold:

- if $t_1 \to t_2$ then $t_1[x/t] \to t_2[x/t]$,
- if $t_1 \succ_i s$ then $t_1[x/t] \succ_i s$.

Recall that $t \succ_i s$ is interpreted as "t is represented by s in (type, state) i". A reduction system is invariant if whenever t is "represented" by s ($t \succ_i s$) and some "reasonable" property holds for s ($us \rightsquigarrow_j s'$), then this same property holds for t ($ut \rightsquigarrow_j s'$), i.e., the system is "invariant" under substitution of t for s.

Lemma 2.3.15. *If a reduction system $\langle \to, \{\succ_i\}_{i\in I}\rangle$ is invariant, then the following condition holds for any terms t, s, s', u.*

- *If $t \leadsto_i s$ and $us \leadsto_j s'$ then $ut \leadsto_j s'$.*

Definition 2.3.16. We define the following reduction systems:

- lambda-calculus with β-reduction: $\lambda\beta = \langle \to_\beta, \emptyset\rangle$,
- lambda-calculus with $\beta\eta$-reduction: $\lambda\beta\eta = \langle \to_{\beta\eta}, \emptyset\rangle$,
- combinatory logic with weak reduction: $\mathrm{CL}w = \langle \to_w, \emptyset\rangle$.

2.3.4 Models

In this section we introduce the notions of combinatory algebra, λ-algebra and λ-model. Combinatory algebras are models of combinatory logic, while λ-algebras and λ-models are models of the lambda-calculus. Our exposition mostly follows [Bar84, Chapter 5].

Definition 2.3.17. A *combinatory algebra* $\mathcal{C}$ is a tuple $\langle C, \cdot, \mathsf{k}, \mathsf{s}\rangle$ where C is a set, $\cdot$ is a binary operation on C, and $\mathsf{k}, \mathsf{s} \in C$ satisfy the following for any $a, b, c \in C$:

- $\mathsf{k} \cdot a \cdot b = a$,
- $\mathsf{s} \cdot a \cdot b \cdot c = a \cdot c \cdot (b \cdot c)$.

The operation $\cdot$ is assumed to be left-associative. We often write $a \in \mathcal{C}$ instead of $a \in C$.

A combinatory algebra $\mathcal{C}$ is *extensional* if all $a, b \in \mathcal{C}$ satisfy:

- $\forall c \in \mathcal{C}(a \cdot c = b \cdot c) \quad \Rightarrow \quad a = b$.

Let $\mathcal{C}$ be a combinatory algebra. A $\mathcal{C}$*-valuation* is a function from the set of variables V to $\mathcal{C}$. Given $t \in \mathbb{T}_{\mathrm{CL}}(\mathcal{C})$ and a $\mathcal{C}$-valuation ρ, we inductively define the value $[\![t]\!]^{\mathcal{C}}_\rho$ of t in $\mathcal{C}$ under ρ:

- $[\![x]\!]^{\mathcal{C}}_\rho = \rho(x)$,
- $[\![\mathsf{K}]\!]^{\mathcal{C}}_\rho = \mathsf{k}$, $[\![\mathsf{S}]\!]^{\mathcal{C}}_\rho = \mathsf{s}$,
- $[\![c]\!]^{\mathcal{C}}_\rho = c$, for $c \in \mathcal{C}$,
- $[\![t_1 t_2]\!]^{\mathcal{C}}_\rho = [\![t_1]\!]^{\mathcal{C}}_\rho \cdot [\![t_2]\!]^{\mathcal{C}}_\rho$.

The superscript $\mathcal{C}$ is dropped when obvious or irrelevant, as is the subscript ρ when t is closed.

A combinatory algebra $\mathcal{C}$ is a λ*-algebra* if for $t, s \in \mathbb{T}_{\mathrm{CL}}(\mathcal{C})$, the condition $t_\lambda =_\beta s_\lambda$ implies that $[\![t]\!]^{\mathcal{C}}_\rho = [\![s]\!]^{\mathcal{C}}_\rho$ for all $\mathcal{C}$-valuations ρ. A combinatory algebra $\mathcal{C}$ is *weakly extensional* when the following conditon holds for any $t, s \in \mathbb{T}_{\mathrm{CL}}(\mathcal{C})$:

- if $[\![t]\!]^{\mathcal{C}}_\rho = [\![s]\!]^{\mathcal{C}}_\rho$ for all ρ, then $[\![\lambda^* x.t]\!]^{\mathcal{C}}_\rho = [\![\lambda^* x.s]\!]^{\mathcal{C}}_\rho$ for all ρ.

A λ*-model* is a weakly extensional λ-algebra.

Note that a weakly extensional combinatory algebra need not be a λ-algebra. Indeed, using [Bar84, Lemma 7.3.5] one may construct a weakly extensional combinatory algebra $\mathcal{C}$ satisfying $[\![((\mathsf{K})_\lambda)_{\mathrm{CL}}]\!]^{\mathcal{C}} \neq [\![\mathsf{K}]\!]^{\mathcal{C}}$. By Lemma 2.3.11 we have $(((\mathsf{K})_\lambda)_{\mathrm{CL}})_\lambda =_\beta (\mathsf{K})_\lambda$, so $\mathcal{C}$ is not a λ-algebra.

Lemma 2.3.18. *If a weakly extensional combinatory algebra $\mathcal{C}$ satisfies $[\![((\mathsf{K})_\lambda)_{\mathrm{CL}}]\!]^{\mathcal{C}} = [\![\mathsf{K}]\!]^{\mathcal{C}}$ and $[\![((\mathsf{S})_\lambda)_{\mathrm{CL}}]\!]^{\mathcal{C}} = [\![\mathsf{S}]\!]^{\mathcal{C}}$, then it is a λ-algebra.*

Proof. Follows from [Bar84, Lemma 5.2.3]. □

Corollary 2.3.19. *Every extensional combinatory algebra is a λ-model.*

2.4 Traditional systems of logic

2.4.1 Propositional logic

In this section we give definitions of the natural deduction systems NJp and NKp of intuitionistic and classical propositional logic. We also define Kripke semantics for NJp and truth-table semantics for NKp. Our exposition mostly follows that of [SU06, Chapter 2].

Definition 2.4.1. The syntax of *propositional formulas* is given by the grammar:

$$\mathcal{F}P \quad ::= \quad V_P \mid \bot \mid \mathcal{F}P \vee \mathcal{F}P \mid \mathcal{F}P \wedge \mathcal{F}P \mid \mathcal{F}P \to \mathcal{F}P$$

where V_P is a set of propositional variables. We use the abbreviation: $\neg\varphi \equiv \varphi \to \bot$.

In what follows, φ, ψ, ν, etc., stand for formulas, Δ, Δ', etc., stand for sets of formulas. The notation Δ, φ abbreviates $\Delta \cup \{\varphi\}$.

A judgement in the system NJp of *intuitionistic propositional logic* has the form $\Delta \vdash \varphi$ where φ is a formula and Δ is a finite set of formulas. The rules of NJp are given in Figure 2.2. For an infinite set of formulas Δ we write $\Delta \vdash \varphi$ if there exists a finite $\Delta' \subseteq \Delta$ such that $\Delta' \vdash \varphi$ is derivable.

The system NKp of *classical propositional logic* is obtained from NJp by replacing the rule ($\bot$E) with:

$$\frac{\Delta, \neg\varphi \vdash \bot}{\Delta \vdash \varphi}\ (\bot\mathrm{E}_c)$$

We write $\Delta \vdash_{\mathrm{NJp}} \varphi$ when $\Delta \vdash \varphi$ is derivable in NJp, and analogously for $\Delta \vdash_{\mathrm{NKp}} \varphi$. The subscript is dropped when obvious from the context.

Definition 2.4.2. A *Kripke* NJp-*model* is a triple $\mathcal{S} = \langle S, \leq, \Vdash \rangle$ where S is a non-empty set of *states*, $\leq$ is a partial order on S, and $\Vdash$ is a binary relation between states and propositional variables which satisfies: if $s \leq s'$ and $s \Vdash p$ then $s' \Vdash p$. We often confuse S with $\mathcal{S}$.

Intuitively, the elements of $\mathcal{S}$ represent states of knowledge. The relation $\leq$ corresponds to extending states by gaining more knowledge, and the relation $\Vdash$ determines which propositional variables are true in a given state.

The relation $\Vdash$ is extended to propositional formulas by the following inductive definition:

- $s \Vdash \varphi \vee \psi$ iff $s \Vdash \varphi$ or $s \Vdash \psi$,
- $s \Vdash \varphi \wedge \psi$ iff $s \Vdash \varphi$ and $s \Vdash \psi$,

$$\frac{}{\Delta, \varphi \vdash \varphi}\ (\mathrm{Ax})$$

$$\frac{\Delta, \varphi \vdash \psi}{\Delta \vdash \varphi \to \psi}\ (\to\mathrm{I}) \qquad \frac{\Delta \vdash \varphi \quad \Delta \vdash \varphi \to \psi}{\Delta \vdash \psi}\ (\to\mathrm{E})$$

$$\frac{\Delta \vdash \varphi \quad \Delta \vdash \psi}{\Delta \vdash \varphi \wedge \psi}\ (\wedge\mathrm{I}) \qquad \frac{\Delta \vdash \varphi \wedge \psi}{\Delta \vdash \varphi}\ (\wedge\mathrm{E}_l) \quad \frac{\Delta \vdash \varphi \wedge \psi}{\Delta \vdash \psi}\ (\wedge\mathrm{E}_r)$$

$$\frac{\Delta \vdash \varphi}{\Delta \vdash \varphi \vee \psi}\ (\vee\mathrm{I}_l) \quad \frac{\Delta \vdash \psi}{\Delta \vdash \varphi \vee \psi}\ (\vee\mathrm{I}_r) \qquad \frac{\Delta \vdash \varphi \vee \psi \quad \Delta, \varphi \vdash \nu \quad \Delta, \psi \vdash \nu}{\Delta \vdash \nu}\ (\vee\mathrm{E})$$

$$\frac{\Delta \vdash \bot}{\Delta \vdash \varphi}\ (\bot\mathrm{E})$$

Figure 2.2: Rules of NJp

- $s \Vdash \varphi \to \psi$ iff $s' \Vdash \psi$ for all $s' \geq s$ with $s' \Vdash \varphi$,
- $s \nVdash \bot$.

The above definition implies the following rule for negation:

- $s \Vdash \neg\varphi$ iff $s' \nVdash \varphi$ for all $s' \geq s$.

Sometimes we write $\mathcal{S}, s \Vdash \varphi$ to make it clear which model is being used. We write $\mathcal{S} \Vdash \varphi$ if $s \Vdash \varphi$ for all $s \in S$. We write $s \Vdash \Delta$ if $s \Vdash \varphi$ for all $\varphi \in \Delta$. Finally, we write $\Delta \Vdash \varphi$ if for every Kripke NJp-model $\mathcal{S}$ and every state s of $\mathcal{S}$, the condition $\mathcal{S}, s \Vdash \Delta$ implies $\mathcal{S}, s \Vdash \varphi$. When we want to emphasize that we are concerned with Kripke NJp-models, we write $\Vdash_{\mathrm{NJp}}$. This will become useful later, when we consider Kripke semantics for various other systems.

Theorem 2.4.3. $\Delta \vdash_{\mathrm{NJp}} \varphi$ *iff* $\Delta \Vdash_{\mathrm{NJp}} \varphi$.

Proof. See e.g. [SU06, Chapter 2]. □

Definition 2.4.4. A *propositional valuation* (or NKp-*valuation*) is a function from V_P to the set $\mathcal{B} = \{0, 1\}$. Valuations will be denoted by u, v, etc. The relation $\models$ between NKp-valuations and propositional formulas is defined inductively:

- $v \models p$ iff $v(p) = 1$,
- $v \models \varphi \vee \psi$ iff $v \models \varphi$ or $v \models \psi$,
- $v \models \varphi \wedge \psi$ iff $v \models \varphi$ and $v \models \psi$,
- $v \models \varphi \to \psi$ iff $v \not\models \varphi$ or $v \models \psi$,
- $v \not\models \bot$.

We write $v \models \Delta$ if $v \models \varphi$ for every $\varphi \in \Delta$. We write $\Delta \models \varphi$ if for every NKp-valuation v such that $v \models \Delta$ we have $v \models \varphi$. When we want to emphasize that the valuations considered are NKp-valuations, we write $\Delta \models_{\mathrm{NKp}} \varphi$.

Note that there is an obvious one-to-one correspondence between single-state Kripke NJp-models and NKp-valuations. Let $\mathcal{S}$ be a single-state Kripke NJp-model. Then the valuation v defined by

$$v(x) = 1 \quad \Leftrightarrow \quad \mathcal{S} \Vdash t$$

satisfies

$$v \models t \quad \Leftrightarrow \quad \mathcal{S} \Vdash t$$

for any term t. Conversely, given an NKp-valuation v, the single-state Kripke model $\mathcal{S}$ defined by

$$\mathcal{S} = \langle \{s_0\}, \{\langle s_0, s_0 \rangle\}, \{\langle s_0, x \rangle \mid v(x) = 1\} \rangle$$

satisfies

$$v \models t \quad \Leftrightarrow \quad \mathcal{S} \Vdash t$$

for any term t.

Theorem 2.4.5. $\Delta \vdash_{\mathrm{NKp}} \varphi$ *iff* $\Delta \models_{\mathrm{NKp}} \varphi$.

2.4.2 First-order predicate logic

In this section we define traditional natural deduction systems NJ and NK of first-order intuitionistic predicate logic and first-order classical predicate logic. Our exposition mostly follows that of [SU06, Chapter 8].

Definition 2.4.6. A signature Σ_{NJ} of NJ consists of function and relation symbols with associated arity. Constants are nullary function symbols. The set of terms $\mathbb{T}_{\mathrm{NJ}}$ of NJ is defined by the grammar:

$$\mathbb{T}_{\mathrm{NJ}} \quad ::= \quad V_{\mathrm{NJ}} \mid f(\mathbb{T}_{\mathrm{NJ}}, \ldots, \mathbb{T}_{\mathrm{NJ}})$$

where f is a function symbol, and V_{NJ} is a set of individual variables. The set of formulas $\mathcal{F}_{\mathrm{NJ}}$ of NJ is defined by:

$$\mathcal{F}_{\mathrm{NJ}} \quad ::= \quad r(\mathbb{T}_{\mathrm{NJ}}, \ldots, \mathbb{T}_{\mathrm{NJ}}) \mid \bot \mid \mathcal{F}_{\mathrm{NJ}} \vee \mathcal{F}_{\mathrm{NJ}} \mid \mathcal{F}_{\mathrm{NJ}} \wedge \mathcal{F}_{\mathrm{NJ}} \mid \mathcal{F}_{\mathrm{NJ}} \to \mathcal{F}_{\mathrm{NJ}} \mid \forall x.\mathcal{F}_{\mathrm{NJ}} \mid \exists x.\mathcal{F}_{\mathrm{NJ}}$$

where r is a relation symbol.

The judgements of NJ have the form $\Delta \vdash \varphi$ where φ is a formula of NJ and Δ is a finite set of formulas. We adopt analogous notational conventions to those in Definition 2.4.1. The rules of NJ are the rules of NJp plus the following.

$$\frac{\Delta \vdash \varphi \quad x \notin \mathrm{FV}(\Delta)}{\Delta \vdash \forall x.\varphi}\ (\forall \mathrm{I}) \qquad \frac{\Delta \vdash \forall x.\varphi}{\Delta \vdash \varphi[x/t]}\ (\forall \mathrm{E})$$

$$\frac{\Delta \vdash \varphi[x/t]}{\Delta \vdash \exists x.\varphi}\ (\exists \mathrm{I}) \qquad \frac{\Delta \vdash \exists x.\varphi \quad \Delta, \varphi \vdash \psi \quad x \notin \mathrm{FV}(\Delta, \psi)}{\Delta \vdash \psi}\ (\exists \mathrm{E})$$

The system NK is obtained from NJ by replacing ($\bot$E) with ($\bot\mathrm{E}_c$) (see Definition 2.4.1). When dealing with NK we write $\mathbb{T}_{\mathrm{NK}}$, $\mathcal{F}_{\mathrm{NK}}$, etc., instead of $\mathbb{T}_{\mathrm{NJ}}$, $\mathcal{F}_{\mathrm{NJ}}$, etc.

Definition 2.4.7. A *classical* NK-*structure* $\mathcal{A} = \langle A, \{f_i^{\mathcal{A}}\}, \{r_i^{\mathcal{A}}\}\rangle$ consists of a non-empty carrier set A, functions $f_i^{\mathcal{A}}$ and relations $r_i^{\mathcal{A}}$ corresponding to function and relation symbols in the signature Σ_{NK}. We often confuse A with $\mathcal{A}$.

An $\mathcal{A}$-*valuation* is a mapping from variables V_{NK} to elements of $\mathcal{A}$. For an $\mathcal{A}$-valuation v and $a \in \mathcal{A}$, the valuation $v[x/a]$ is defined as the valuation u such that $u(x) = a$ and $u(y) = v(y)$ for $y \neq x$. Given an $\mathcal{A}$-valuation v we define the *value* $[\![t]\!]_v^{\mathcal{A}}$ *of a term* $t \in \mathbb{T}_{\mathrm{NK}}$ by induction:

- $[\![x]\!]_v^{\mathcal{A}} = v(x)$,
- $[\![f(t_1, \ldots, t_n)]\!]_v^{\mathcal{A}} = f^{\mathcal{A}}([\![t_1]\!]_v^{\mathcal{A}}, \ldots, [\![t_n]\!]_v^{\mathcal{A}})$,

where $f^{\mathcal{A}}$ is the function in $\mathcal{A}$ corresponding to the function symbol f.

For a formula φ the relation $\mathcal{A}, v \models \varphi$ of satisfaction is defined inductively:

- $\mathcal{A}, v \models r(t_1, \ldots, t_n)$ iff $r^{\mathcal{A}}([\![t_1]\!]_v^{\mathcal{A}}, \ldots, [\![t_n]\!]_v^{\mathcal{A}})$ holds,
- $\mathcal{A}, v \not\models \bot$,
- $\mathcal{A}, v \models \varphi \lor \psi$ iff $\mathcal{A}, v \models \varphi$ or $\mathcal{A}, v \models \psi$,
- $\mathcal{A}, v \models \varphi \land \psi$ iff $\mathcal{A}, v \models \varphi$ and $\mathcal{A}, v \models \psi$,
- $\mathcal{A}, v \models \varphi \to \psi$ iff $\mathcal{A}, v \models \varphi$ implies $\mathcal{A}, v \models \psi$,
- $\mathcal{A}, v \models \forall x.\varphi$ iff for every $a \in \mathcal{A}$ we have $\mathcal{A}, v[x/a] \models \varphi$,
- $\mathcal{A}, v \models \exists x.\varphi$ iff there exists $a \in \mathcal{A}$ such that $\mathcal{A}, v[x/a] \models \varphi$.

We write $\mathcal{A} \models \varphi$ if $\mathcal{A}, v \models \varphi$ for every v. We write $\mathcal{A} \models \Delta$ ($\mathcal{A}, v \models \Delta$) if $\mathcal{A} \models \varphi$ ($\mathcal{A}, v \models \varphi$) for every $\varphi \in \Delta$. Finally, we write $\Delta \models \varphi$ if for every $\mathcal{A}$ and v such that $\mathcal{A}, v \models \Delta$ we have $\mathcal{A}, v \models \varphi$. We sometimes use $\models_{\mathrm{NK}}$ instead of $\models$ to emphasize which system we have in mind.

A structure $\mathcal{B} = \langle B, \{f_i^{\mathcal{B}}\}, \{r_i^{\mathcal{B}}\}\rangle$ is an *extension* of $\mathcal{A} = \langle A, \{f_i^{\mathcal{A}}\}, \{r_i^{\mathcal{A}}\}\rangle$, denoted $\mathcal{A} \subseteq \mathcal{B}$, if the following hold:

- $A \subseteq B$,
- $r_i^{\mathcal{A}} \subseteq r_i^{\mathcal{B}}$ for all i,
- $f_i^{\mathcal{A}}(a) = f_i^{\mathcal{B}}(a)$ for $a \in A$, for all i.

Theorem 2.4.8. $\Delta \vdash_{\mathrm{NK}} \varphi$ *iff* $\Delta \models_{\mathrm{NK}} \varphi$.

Proof. See e.g. [SU06, Theorem 8.4.7]. □

Definition 2.4.9. A *Kripke* NJ-*model* is a triple $\mathcal{S} = \langle S, \leq, \{\mathcal{A}_s \mid s \in S\}\rangle$ where S is a non-empty set of states, $\leq$ is a partial order on states, and the $\mathcal{A}_s$ are classical structures such that: $s \leq s'$ implies $\mathcal{A}_s \subseteq \mathcal{A}_{s'}$.

Let φ be a formula and v be an $\mathcal{A}_s$-valuation. Note that then v is an $\mathcal{A}_{s'}$-valuation for all $s' \geq s$. The relation $s, v \Vdash \varphi$ is defined by induction on φ.

- $s, v \Vdash r(t_1, \ldots, t_n)$ iff $r^{\mathcal{A}_s}([\![t_1]\!]_v^{\mathcal{A}_s}, \ldots, [\![t_n]\!]_v^{\mathcal{A}_s})$ holds,
- $s, v \nVdash \bot$,

- $s, v \Vdash \varphi \vee \psi$ iff $s, v \Vdash \varphi$ or $s, v \Vdash \psi$,
- $s, v \Vdash \varphi \wedge \psi$ iff $s, v \Vdash \varphi$ and $s, v \Vdash \psi$,
- $s, v \Vdash \varphi \to \psi$ iff for all $s' \geq s$ such that $s', v \Vdash \varphi$ we have $s', v \Vdash \psi$,
- $s, v \Vdash \forall x.\varphi$ iff for all $s' \geq s$ and all $a \in \mathcal{A}_{s'}$ we have $s', v[x/a] \Vdash \varphi$,
- $s, v \Vdash \exists x.\varphi$ iff there exists $a \in \mathcal{A}_s$ such that $s, v[x/a] \Vdash \varphi$.

The symbol $\Vdash$ is used as usual (see Definition 2.4.2). In particular, $\Delta \Vdash \varphi$ means that for all Kripke NJ-models $\mathcal{S}$, all $s \in \mathcal{S}$ and all $\mathcal{A}_s$-valuations v, if $s, v \Vdash \Delta$ then $s, v \Vdash \varphi$.

Theorem 2.4.10. $\Delta \vdash_{\mathrm{NJ}} \varphi$ *iff* $\Delta \Vdash_{\mathrm{NJ}} \varphi$.

Proof. See e.g. [SU06, Theorem 8.6.7]. □

2.4.3 Higher-order predicate logic

In this section we present the system NKω of classical higher-order logic, together with some of its variants. For more background on higher-order logic see e.g. [BBK04, Lei94, Chu40].

Definition 2.4.11. The system NKω of *intensional classical higher-order logic* is defined as follows.

- The *types* of NKω are given by

$$\mathscr{T} \quad ::= \quad o \mid \mathcal{B} \mid \mathscr{T} \to \mathscr{T}$$

  where $\mathcal{B}$ is a specific finite set of base types. The type o is the type of propositions. We assume $o \notin \mathcal{B}$.

- The set of terms of NKω of type τ, denoted T_τ, is defined as follows:
  - $V_\tau, \Sigma_\tau \subseteq T_\tau$ where V_τ is the set of variables of type τ and Σ_τ is the set of constants of type τ,
  - if $t_1 \in T_{\sigma \to \tau}$ and $t_2 \in T_\sigma$ then $t_1 t_2 \in T_\tau$,
  - if $x \in V_{\tau_1}$ and $t \in T_{\tau_2}$ then $\lambda x : \tau_1 \,.\, t \in T_{\tau_1 \to \tau_2}$,
  - if $\varphi, \psi \in T_o$ then $\varphi \to \psi \in T_o$,
  - if $x \in V_\tau$ and $\varphi \in T_o$ then $\forall x : \tau \,.\, \varphi \in T_o$,

  where for each type τ the set V_τ is a countable set of variables and Σ_τ is a countable set of constants. We assume that the sets V_τ and Σ_σ are all pairwise disjoint. Terms of type o are *formulas*. As usual, we omit spurious brackets and assume that application associates to the left. We identify α-equivalent terms, i.e., terms differing only in the names of bound variables are considered identical.

- The rules of NKω are given in Figure 2.3, where Δ is a finite set of formulas, φ, ψ are formulas, and $\bot \equiv \forall p : o \,.\, p$. The notation Δ, φ is a shorthand for $\Delta \cup \{\varphi\}$.

$$\frac{}{\Delta, \varphi \vdash \varphi}\ (\mathrm{Ax}) \qquad \frac{\Delta, \varphi \to \bot \vdash \bot}{\Delta \vdash \varphi}\ (\bot\mathrm{E}_c)$$

$$\frac{\Delta, \varphi \vdash \psi}{\Delta \vdash \varphi \to \psi}\ (\to\mathrm{I}) \qquad \frac{\Delta \vdash \varphi \to \psi \quad \Delta \vdash \varphi}{\Delta \vdash \psi}\ (\to\mathrm{E})$$

$$\frac{\Delta \vdash \varphi \quad x \notin \mathrm{FV}(\Delta), x \in V_\tau}{\Delta \vdash \forall x : \tau \,.\, \varphi}\ (\forall\mathrm{I}) \qquad \frac{\Delta \vdash \forall x : \tau \,.\, \varphi \quad t \in T_\tau}{\Delta \vdash \varphi[x/t]}\ (\forall\mathrm{E})$$

$$\frac{\Delta \vdash \varphi \quad \varphi =_{\beta\eta} \psi}{\Delta \vdash \psi}\ (\mathrm{conv})$$

Figure 2.3: Rules of NKω

In NKω, we define Leibniz equality in type $\tau \in \mathscr{T}$ by

$$t_1 =_\tau t_2 \equiv \forall p : \tau \to o \,.\, pt_1 \to pt_2$$

The system NKω is intensional. An extensional variant eNKω may be obtained by adding the following axioms for all $\tau, \sigma \in \mathscr{T}$:

$$e_f : \forall f_1, f_2 : \tau \to \sigma \,.\, (\forall x : \tau \,.\, f_1 x =_\sigma f_2 x) \to (f_1 =_{\tau\to\sigma} f_2)$$

$$e_b : \forall \varphi_1, \varphi_2 : o \,.\, ((\varphi_1 \to \varphi_2) \land (\varphi_2 \to \varphi_1)) \to (\varphi_1 =_o \varphi_2)$$

For an arbitrary set of formulas Δ we write $\Delta \vdash_S \varphi$ if φ is derivable from a subset of Δ in system S. The subscript is dropped when obvious or irrelevant.

The only logical connectives in NKω are $\to$ and $\forall$. The remaining connectives may be defined as follows:

$$\begin{aligned} \bot &\equiv \forall p : o \,.\, p \\ \neg\varphi &\equiv \varphi \to \bot \\ \varphi \land \psi &\equiv \forall p : o \,.\, (\varphi \to \psi \to p) \to p \\ \varphi \lor \psi &\equiv \forall p : o \,.\, (\varphi \to p) \to (\psi \to p) \to p \\ \exists x : \tau \,.\, \varphi &\equiv \forall p : o \,.\, (\forall x : \tau \,.\, \varphi \to p) \to p \end{aligned}$$

Lemma 2.4.12. *The rules in Figure 2.4 are admissible in* NKω*.*

Definition 2.4.13 (Standard semantics)**.** A *standard model* is a tuple

$$\mathcal{M} = \langle \{\mathcal{D}_\tau \mid \tau \in \mathscr{T}\}, I \rangle$$

where each $\mathcal{D}_\tau$ is a non-empty set for $\tau \in \mathcal{B}$, $\mathcal{D}_o = \{\top, \bot\}$, each $\mathcal{D}_{\tau_1\to\tau_2}$ is the set of all functions from $\mathcal{D}_{\tau_1}$ to $\mathcal{D}_{\tau_2}$, and I is a function mapping constants of type τ to $\mathcal{D}_\tau$. We assume that $\mathcal{D}_{\tau_1} \cap \mathcal{D}_{\tau_2} = \emptyset$ for $\tau_1, \tau_2 \in \mathcal{B}$, $\tau_1 \neq \tau_2$.

An $\mathcal{M}$-*valuation* is a function mapping variables of type τ to $\mathcal{D}_\tau$. Given an $\mathcal{M}$-valuation ρ, the interpretation function $[\![]\!]^{\mathcal{M}}_\rho$, mapping each term $t \in T_\tau$ to $\mathcal{D}_\tau$, for each $\tau \in \mathscr{T}$, is defined inductively:

$$\frac{\Delta \vdash \varphi \quad \Delta \vdash \psi}{\Delta \vdash \varphi \wedge \psi} \ (\wedge\mathrm{I}) \qquad \frac{\Delta \vdash \varphi \wedge \psi}{\Delta \vdash \varphi} \ (\wedge\mathrm{E}_l) \quad \frac{\Delta \vdash \varphi \wedge \psi}{\Delta \vdash \psi} \ (\wedge\mathrm{E}_r)$$

$$\frac{\Delta \vdash \varphi}{\Delta \vdash \varphi \vee \psi} \ (\vee\mathrm{I}_l) \quad \frac{\Delta \vdash \psi}{\Delta \vdash \varphi \vee \psi} \ (\vee\mathrm{I}_r) \qquad \frac{\Delta \vdash \varphi \vee \psi \quad \Delta, \varphi \vdash \nu \quad \Delta, \psi \vdash \nu}{\Delta \vdash \nu} \ (\vee\mathrm{E})$$

$$\frac{\Delta \vdash \varphi[x/t] \quad t \in T_\tau}{\Delta \vdash \exists x : \tau \,.\, \varphi} \ (\exists\mathrm{I}) \qquad \frac{\Delta \vdash \exists x : \tau \,.\, \varphi \quad \Delta, \varphi \vdash \psi \quad x \notin \mathrm{FV}(\Delta, \psi)}{\Delta \vdash \psi} \ (\exists\mathrm{E})$$

$$\frac{\Delta \vdash \bot}{\Delta \vdash \varphi} \ (\bot\mathrm{E})$$

Figure 2.4: Admissible rules in NKω

- $[\![x]\!]^{\mathcal{M}}_{\rho} = v(x)$,
- $[\![c]\!]^{\mathcal{M}}_{\rho} = I(c)$,
- $[\![t_1 t_2]\!]^{\mathcal{M}}_{\rho} = [\![t_1]\!]^{\mathcal{M}}_{\rho}([\![t_2]\!]^{\mathcal{M}}_{\rho})$,
- $[\![\lambda x.t]\!]^{\mathcal{M}}_{\rho}(d) = [\![t]\!]^{\mathcal{M}}_{\rho[x/d]}$ for $d \in \mathcal{D}_{\tau_1}$, where $x \in V_{\tau_1}$ and $t \in T_{\tau_2}$,
- $[\![\varphi \to \psi]\!]^{\mathcal{M}}_{\rho} = \top$ iff $[\![\varphi]\!]^{\mathcal{M}}_{\rho} = \bot$ or $[\![\psi]\!]^{\mathcal{M}}_{\rho} = \top$,
- $[\![\forall x : \tau \,.\, \varphi]\!]^{\mathcal{M}}_{\rho} = \top$ iff for all $d \in \mathcal{D}_\tau$ we have $[\![\varphi]\!]^{\mathcal{M}}_{\rho[x/d]} = \top$.

The satisfaction relation $\models_{\mathrm{std}}$ is defined in the standard way.

Theorem 2.4.14. *If $\Delta \vdash_{e\mathrm{NK}\omega} \varphi$ then $\Delta \models_{\mathrm{std}} \varphi$.*

Of course, eNKω is not complete with respect to standard semantics. There are other notions of models with respect to which various systems of higher-order logic are complete. See e.g. [BBK04].

Chapter 3

Paradoxes

In this chapter we present three paradoxes in systems of illative combinatory logic: Curry's paradox, Bunder's paradox, and the Kleene-Rosser paradox. These paradoxes show certain limitations on the rules an illative system may contain. To be able to formulate the paradoxes in a general way, we now give a definition of a general illative system.

Definition 3.1. A *general $\lambda\beta\eta$-illative system* (resp. $\lambda\beta$- or CLw-illative system) is a pair $\mathcal{I} = \langle \Sigma, \vdash \rangle$ where Σ is a set of constants and $\vdash$ is a binary *provability relation* between finite sets of terms from $\mathbb{T}$ and a term from $\mathbb{T}$, where $\mathbb{T} = \mathbb{T}_\lambda(\Sigma)$ for a $\lambda\beta\eta$- or $\lambda\beta$-illative system, and $\mathbb{T} = \mathbb{T}_{\mathrm{CL}}(\Sigma)$ for a CLw-illative system. A *(general) illative system* is a general $\lambda\beta\eta$-, $\lambda\beta$- or a CLw-illative system. Conventions from Section 1.1 apply. In particular, the equality $=$ is used to generically denote $\beta\eta$-, β- or CLw-equality, depending on the kind of illative system considered. We sometimes write $\vdash_\mathcal{I}$ for the provability relation of an illative system $\mathcal{I}$. We say that an illative system $\mathcal{I}$ *contains* illative primitives $P_1, \ldots, P_n$ and rules (axioms) $R_1, \ldots, R_m$ if there are terms $X_1, \ldots, X_n$ of $\mathcal{I}$ such that all rules (axioms) $R_1, \ldots, R_m$ are true when P_i is interpreted with X_i and $\vdash$ with $\vdash_\mathcal{I}$. We do not give a completely precise definition of a rule or "interpretation" of an illative primitive, because the meaning is intuitively obvious and precise definitions would only add excessive formalism.

3.1 Curry's paradox

In this section we present Curry's paradox. It was first obtained in [Cur42b]. See also [CFC58, §8A] and [Sel09].

Theorem 3.1.1 (Curry's paradox)**.** *Any illative system $\mathcal{I}$ containing the illative primitive* P*, the axiom* (Ax) *and the rules* (DED)*,* (PE)*,* (Eq) *below is inconsistent, i.e.,* $\vdash_\mathcal{I} Y$ *for an arbitrary term* Y*.*

$$\frac{}{\Gamma, X \vdash X}\ (\mathrm{Ax}) \qquad\qquad \frac{\Gamma \vdash X \quad X = Y}{\Gamma \vdash Y}\ (\mathrm{Eq})$$

$$\frac{\Gamma, X \vdash Y}{\Gamma \vdash X \supset Y}\ (\mathrm{DED}) \qquad\qquad \frac{\Gamma \vdash X \supset Y \quad \Gamma \vdash X}{\Gamma \vdash Y}\ (\mathsf{PE})$$

Proof. Let Y be an arbitrary term. Define a term X by the equation $X = (X \supset Y)$ (see Section 2.3.2). We have:

1. $X \vdash X$ by (Ax),
2. $X \vdash X \supset Y$ by 1 and (Eq),
3. $X \vdash Y$ by 2, 1 and (PE),
4. $\vdash X \supset Y$ by 3 and (DED),
5. $\vdash Y$ by 4, 1 and (PE).

□

The principle of combinatory completeness states that any function we can define intuitively by means of a variable can be represented formally as an entity of the system (cf. [CFC58, p. 5]). More precisely, for any term M and any variable x, there should exist a term X such that $x \notin \mathrm{FV}(X)$ and $Xx = M$. Curry's paradox shows that this principle (which necessitates the rule (Eq) if fundamental properties of equality are to be retained) is incompatible with deductive completeness (rule (DED)). Therefore, if we want to retain the rule (Eq), some restrictions to (DED) are necessary.

3.2 Bunder's paradox

Bunder's paradox shows a limitation on those illative systems which use the illative primitive H to restrict the implication introduction rule. Essentially, any (reasonable) sufficiently strong illative system with an axiom scheme $\vdash \mathsf{H}^k X$, with X an arbitrary term, is inconsistent, where $\mathsf{H}^k X$ denotes k-time application of H to X, e.g., $\mathsf{H}^3 X \equiv \mathsf{H}(\mathsf{H}(\mathsf{H}X))$. In what follows we adopt the convention $\mathsf{H}^0 X \equiv X$.

Theorem 3.2.1 (Bunder's paradox)**.** *Any illative system $\mathcal{I}$ containing the illative primitives* P*,* H*, the following rules* (PE)*,* (PI)*,* (HI)*,* (Eq) *and the axioms* (Ax) *and* (H^k) *for some $k > 0$ is inconsistent, i.e., $\vdash_{\mathcal{I}} Y$ for an arbitrary term Y.*

$$\frac{}{\Gamma, X \vdash X}\ (\mathrm{Ax}) \qquad \frac{}{\Gamma \vdash \mathsf{H}^k X}\ (\mathsf{H}^k)$$

$$\frac{\Gamma, X \vdash Y \quad \Gamma \vdash \mathsf{H}X}{\Gamma \vdash X \supset Y}\ (\mathsf{PI}) \qquad \frac{\Gamma \vdash X \supset Y \quad \Gamma \vdash X}{\Gamma \vdash Y}\ (\mathsf{PE})$$

$$\frac{\Gamma \vdash X}{\Gamma \vdash \mathsf{H}X}\ (\mathsf{HI}) \qquad \frac{\Gamma \vdash X \quad X = Y}{\Gamma \vdash Y}\ (\mathrm{Eq})$$

Proof. Let Y be an arbitrary term. Define a term X by the equation

$$X = (\mathsf{H}^{k-1} X \supset \ldots \supset \mathsf{H}^2 X \supset \mathsf{H}X \supset X \supset Y)$$

where $\supset$ is assumed to associate to the right. We have

$$\mathsf{H}^{k-1}X, \ldots, \mathsf{H}X, X \vdash X$$

by (Ax), and thus

$$\mathsf{H}^{k-1}X, \ldots, \mathsf{H}X, X \vdash \mathsf{H}^{k-1}X \supset \ldots \supset \mathsf{H}X \supset X \supset Y$$

by (Eq). Since

$$\mathsf{H}^{k-1}X, \ldots, \mathsf{H}X, X \vdash \mathsf{H}^{i}X$$

for $1 \leq i \leq k-1$ by (Ax), we obtain

$$\mathsf{H}^{k-1}X, \ldots, \mathsf{H}X, X \vdash Y$$

by applying (PE) with $\mathsf{H}^{k-1}X, \ldots, \mathsf{H}X, X$ consecutively. Now, applying (PI) consecutively $k-1$ times we obtain

$$\mathsf{H}^{k-1}X \vdash \mathsf{H}^{k-2}X \supset \ldots \supset \mathsf{H}X \supset X \supset Y.$$

Since $\vdash \mathsf{H}^{k}X$ by (H^{k}), i.e. $\vdash \mathsf{H}(\mathsf{H}^{k-1}X)$, we have

$$\vdash \mathsf{H}^{k-1}X \supset \ldots \supset \mathsf{H}X \supset X \supset Y$$

by (PI), so

$$\vdash X$$

by (Eq). Using (HI) we may now obtain $\vdash \mathsf{H}^{i}X$ for $1 \leq i \leq k-1$. Since

$$\vdash \mathsf{H}^{k-1}X \supset \ldots \supset \mathsf{H}X \supset X \supset Y$$

we ultimately obtain

$$\vdash Y$$

by applying (PE) consecutively with $\mathsf{H}^{k-1}X, \ldots, \mathsf{H}X, X$. □

For $k = 2$ a variant of the above result was shown by Curry in [Cur42c, Cur42a], but under somewhat different assumptions. Curry also stated that the result holds for arbitrary $k > 0$ but the proof was lost. The result was later rediscovered by Bunder [Bun70]. See also [CHS72, §15C5]. The proof in [Bun70] does not apply to the system $\mathcal{F}_{21}^{*}$ from [CHS72] and requires the following rule (PHI).

$$\frac{\Gamma, X \vdash \mathsf{H}Y \quad \Gamma \vdash \mathsf{H}X}{\Gamma \vdash \mathsf{H}(X \supset Y)} \ (\mathsf{PHI})$$

In [Bun76] the inconsistency of $\mathcal{F}_{21}^{*}$ is shown by adapting the method of [Bun70]. Earlier in [Bun74b] Bunder proves inconsistency of a related system of Seldin [Sel68]. In [BM78] Bunder and Meyer extend the results of [Bun70, Bun76] to systems similar to $\mathcal{F}_{21}^{*}$, which includes the result of our Theorem 3.2.1. Actually, in [BM78] Bunder and Meyer essentially show inconsistency of any system with (Ax), (H^k), (Eq), (PE), (HI), the rule

$$\frac{\Gamma \vdash \mathsf{H}X \quad \Gamma \vdash \mathsf{H}Y}{\Gamma \vdash \mathsf{H}(X \supset Y)} \ (\mathsf{PHI}')$$

and with

$$\frac{\Gamma, X \vdash Y \quad \Gamma \vdash \mathsf{H}X \quad \Gamma \vdash \mathsf{H}Y}{\Gamma \vdash X \supset Y} \ (\mathsf{PI}')$$

instead of (PI), and then they note that the inconsistency of systems without (PHI′) and with (PI) instead of (PI′) follows by an analogous argument. In the USSR, Shumikhin [Shu78] seems to have discovered a proof of Theorem 3.2.1 independently of [Bun76, BM78] (but he knew about [Bun74b] which he cites). Our proof of Theorem 3.2.1 follows [BM78, Shu78]. We chose to name the paradox after Bunder, since he seems to be the person who contributed most to its discovery.

Bunder's paradox shows that (HI) and (H^k) are incompatible. The rule (HI) seems very natural – it says that if X is provable (it is true) then it is a proposition. On the other hand, the interpretation of (H^k) is less clear. Hence, it is usually the choice to abandon (H^k) in favor of (HI). This choice is also adopted in the illative systems studied in the present work.

3.3 Kleene-Rosser paradox

The first paradox in the early illative systems of Church [Chu32, Chu33] and Curry [Cur30, Cur31, Cur32, Cur33, Cur34b] was derived by Kleene and Rosser [KR35]. The Kleene-Rosser paradox is much more complicated than the subsequently discovered Curry's paradox [Cur42b]. However, it may be adapted to apply to some systems to which Curry's paradox does not apply. In this section we present a variant of the Kleene-Rosser paradox.

The original paper [KR35] of Kleene and Rosser is very dense and presupposes intimate knowledge of some specific illative systems and of a few previous papers. A more readable but still quite complex exposition may be found in [Cur41b]. The method of deriving the paradox in [Cur41b] is different from the original method of [KR35] and stronger assumptions are used. In particular, both deductive completeness for Ξ, i.e., essentially the rule

$$\frac{\Gamma, Xx \vdash Yx \quad x \notin \mathrm{FV}(\Gamma, X, Y)}{\Gamma \vdash \Xi XY} \ (\Xi\mathrm{DED})$$

and full combinatory completeness are presupposed, which makes the system amenable to the much simpler Curry's paradox[1]. The systems of Church [Chu32, Chu33] shown inconsistent in [KR35] restrict both the combinatory completeness (they are based on the λI-calculus) and deductive completeness (essentially by adding in (ΞDED) a premise $\Gamma \vdash XZ$ with Z some

[1]In an illative system with (ΞDED) and (Eq), the illative primitive P may be defined by $\mathsf{P} \equiv \lambda x.\Xi(\mathsf{K}x)(\mathsf{K}x)$. Then (DED) follows from (ΞDED) and (Eq).

arbitrary term). Thus Curry's paradox cannot be directly derived in them. See also [CHS72, p. 182] and [Bun73b].

Our presentation of the Kleene-Rosser paradox is loosely based on [Cur41b], but differs from that paper considerably in technical details, and also in the assumptions about the illative system. Our exposition is simpler than [Cur41b], but we assume more illative primitives and rules. In particular, we assume an unrestricted induction principle. In [Cur41b] a similar principle is derived using deductive completeness. On the other hand, we do not presuppose any form of deductive completeness – neither (DED), (ΞDED) nor even the weak deductive completeness assumed in the original proof of Kleene and Rosser. Our presentation of the Kleene-Rosser paradox reveals an essential incompatibility between an unrestricted induction principle and a Hilbert-style formulation of an illative system.

The Kleene-Rosser paradox essentially refines the Richard paradox by setting it up formally inside an illative system. The Richard paradox may be informally described as follows. The set of definable numerical functions (i.e. functions from $\mathbb{N}$ to $\mathbb{N}$) is countable, because each such function is defined by a sentence in the language, i.e., by a finite sequence of symbols. Let $(f_i)_{i\in\mathbb{N}}$ be an enumeration of all definable numerical functions. Define a function f by $f(n) = f_n(n) + 1$ for $n \in \mathbb{N}$. Since f is definable, there exists $m \in \mathbb{N}$ such that $f_m = f$. But then $f_m(m) = f(m) = f_m(m) + 1$, so $0 = 1$. Contradiction.

The above argument is made more precise in the following proposition, where N represents the type of natural numbers, s represents the successor function, $\mathcal{U}$ represents an enumeration of definable numerical functions ($\mathcal{U}Y = X$ means that Y represents the number of the term X, and X represents a numerical function), and F is the functionality combinator (see Section 1.1).

Proposition 3.3.1 (Richard paradox)**.** *Any illative system $\mathcal{I}$ which contains the illative primitives N, s, $\mathcal{U}$, F and satisfies the conditions (a) – (g) below, is inconsistent, i.e., $\vdash_{\mathcal{I}} Y$ for an arbitrary term Y. In the following conditions X, Y are arbitrary terms and Γ is an arbitrary finite set of terms.*

(a) $X \vdash X$.

(b) *If* $\mathsf{N}x \vdash \mathsf{N}X$ *then* $\vdash \mathsf{FNN}(\lambda x.X)$.

(c) *If* $\Gamma \vdash \mathsf{FN}XY$ *and* $\Gamma \vdash \mathsf{N}Z$ *then* $\Gamma \vdash X(YZ)$.

(d) $\Gamma \vdash \mathsf{FNNs}$.

(e) *If* $\vdash \mathsf{N}X$ *and* $X = \mathsf{s}X$ *then* $\vdash Y$.

(f) *If* $\vdash \mathsf{FNN}X$ *then there is* Y *with* $\vdash \mathsf{N}Y$ *and* $\mathcal{U}Y = X$.

(g) $\Gamma \vdash \mathsf{FN(FNN)}\mathcal{U}$.

Proof. Let Y be an arbitrary term. Define $M \equiv \lambda x.\mathsf{s}(\mathcal{U}xx)$. We have:

1. $\mathsf{N}x \vdash \mathsf{N}x$ by (a),
2. $\mathsf{N}x \vdash \mathsf{FNN}(\mathcal{U}x)$ by (c), (g) and 1,
3. $\mathsf{N}x \vdash \mathsf{N}(\mathcal{U}xx)$ by (c), 2 and 1,

4. $\mathsf{N}x \vdash \mathsf{N}(\mathsf{s}(\mathcal{U}xx))$ by (c), (d) and 3,
5. $\vdash \mathsf{FNN}M$ by (b) and 4,
6. $\vdash \mathsf{N}X$ and $\mathcal{U}X = M$ for some X, by (f) and 5,
7. $\vdash \mathsf{N}(MX)$ by (c), 5 and 6,
8. $MX = \mathsf{s}(MX)$ by 6,
9. $\vdash Y$ by (e), 7 and 8.

□

The conditions $(a) - (e)$ are very natural and we would expect them to hold in any illative system containing the type of natural numbers and the functionality combinator. Also, in any reasonable illative system with the type of natural numbers represented by N, and with a recursively enumerable set of theorems, there is a term $\mathcal{U}$ satisfying (f), thanks to unrestricted recursive definitions available in lambda-calculus and combinatory logic. Indeed, assuming enough operations on terms representing natural numbers, because the set of theorems is recursively enumerable, one may construct a term Ω which enumerates (possibly with repetitions) the terms representing the numerical codes of terms X satisfying $\vdash \mathsf{FNN}X$, i.e., a term Ω such that for every term X satisfying $\vdash \mathsf{FNN}X$ there is $n \in \mathbb{N}$ with $\Omega\underline{n} = \underline{m}$ where $m \in \mathbb{N}$ is the code of X, and $\underline{n}, \underline{m}$ are terms representing the numbers n and m, respectively. With an appropriate coding scheme it is also not difficult to construct a term T which "evaluates" terms representing numerical codes of terms, i.e., $\mathsf{T}\underline{n} = X$ when $n \in \mathbb{N}$ is the code of X and $\underline{n}$ represents the number n. Then we may take $\mathcal{U} \equiv \mathsf{T} \circ \Omega$. Therefore, the real problem is with condition (g), which states that $\mathcal{U}$ may be typed inside the system. In the remainder of this section we shall formulate some seemingly innocuous assumptions on an illative system, which nonetheless will be shown to imply (g).

Definition 3.3.2. We define *numerals* by:

$$\begin{aligned} \underline{0} &\equiv \mathsf{I} \\ \underline{n+1} &\equiv \lambda x.x(\mathsf{KI})\underline{n} \end{aligned}$$

See also [Bar84, Chapter 6]. We define the following terms:

$$\begin{aligned} \mathsf{s} &\equiv \lambda nx.x(\mathsf{KI})n \\ \mathsf{p} &\equiv \lambda n.n(\mathsf{KI}) \\ \mathsf{z} &\equiv \lambda n.n\mathsf{K} \end{aligned}$$

An illative system *contains arithmetic* if it contains the illative primitives P, H, Ξ, N, Q and the rules from Figure 3.1.

The primitive Q represents equality on natural numbers. The primitive N represents the type of natural numbers. The rule (NInd) expresses an unrestricted induction principle – nothing is assumed a priori about the term X. The term s represents the successor function on natural numbers, p the predecessor, and z the test for zero.

$$\frac{}{\Gamma, X \vdash X}\ (\text{Ax}) \qquad \frac{\Gamma \vdash X}{\Gamma, Y \vdash X}\ (\text{Weak}) \qquad \frac{\Gamma \vdash X \quad \Gamma, X \vdash Y}{\Gamma \vdash Y}\ (\text{Cut})$$

$$\frac{\Gamma \vdash X \quad X = Y}{\Gamma \vdash Y}\ (\text{Eq})$$

$$\frac{\Gamma, X \vdash Y \quad \Gamma \vdash \mathsf{H}X}{\Gamma \vdash X \supset Y}\ (\mathsf{PI}) \qquad \frac{\Gamma \vdash X \supset Y \quad \Gamma \vdash X}{\Gamma \vdash Y}\ (\mathsf{PE})$$

$$\frac{\Gamma, \mathsf{N}x \vdash Xx \quad x \notin \mathrm{FV}(\Gamma, X)}{\Gamma \vdash \Xi\mathsf{N}X}\ (\Xi\mathsf{NI}) \qquad \frac{\Gamma \vdash \Xi\mathsf{N}X \quad \Gamma \vdash \mathsf{N}Y}{\Gamma \vdash XY}\ (\Xi\mathsf{NE})$$

$$\frac{\Gamma \vdash X\underline{0} \quad \Gamma, \mathsf{N}x, Xx \vdash X(\mathsf{s}x) \quad x \notin \mathrm{FV}(\Gamma, X)}{\Gamma \vdash \Xi\mathsf{N}X}\ (\mathsf{NInd})$$

$$\frac{}{\Gamma \vdash \mathsf{N}\underline{0}}\ (\mathsf{NI}_0) \qquad \frac{\Gamma \vdash \mathsf{N}X}{\Gamma \vdash \mathsf{N}(\mathsf{s}X)}\ (\mathsf{NI}_s)$$

$$\frac{\Gamma \vdash \mathsf{N}X}{\Gamma \vdash \mathsf{Q}XX}\ (\mathsf{QI}) \qquad \frac{\Gamma \vdash \mathsf{Q}XY \quad \Gamma \vdash ZX}{\Gamma \vdash ZY}\ (\mathsf{QE}) \qquad \frac{\Gamma \vdash \mathsf{N}X \quad \Gamma \vdash \mathsf{N}Y}{\Gamma \vdash \mathsf{H}(\mathsf{Q}XY)}\ (\mathsf{QH})$$

Figure 3.1: Rules for illative systems containing arithmetic

In what follows we shall implicitly assume a fixed illative system containing arithmetic. Unless otherwise stated, all the following lemmas concern illative systems containing arithmetic. The symbols Γ, Γ', etc., are used to denote finite sets of terms, and X, Y, etc., denote terms, unless otherwise specified.

The results of this section do not depend on the details of the encoding of natural numbers. In fact, we use only the properties of numerals and the terms s, p and z summmarised in the following lemma. Note, however, that we could not use Church numerals, because the second point of the lemma would not hold.

Lemma 3.3.3. *We have the following equalities:*[2]

1. $\mathsf{s}\,\underline{n} = \underline{n+1}$,
2. $\mathsf{p}(\mathsf{s}X) = X$ *for an arbitrary term* X,
3. $\mathsf{z}\,\underline{0} = \mathsf{K}$,
4. $\mathsf{z}(\mathsf{s}X) = \mathsf{KI}$ *for an arbitrary term* X.

We use the notations $\bot \equiv \mathsf{Q}\,\underline{0}\,\underline{1}$ and $\top \equiv \mathsf{Q}\,\underline{0}\,\underline{0}$. We also write $\neg X$ for $X \supset \bot$. Like in Section 1.1 we use the notation $\mathsf{F} \equiv \lambda fxy.\Xi x(\lambda z.y(fz))$.

Lemma 3.3.4. *The rules from Figure 3.2 are admissible.*

[2]As usual = denotes either weak-, β- or $\beta\eta$-equality, depending on the kind of the illative system considered.

$$\frac{\Gamma \vdash \mathsf{N}X \quad \Gamma \vdash \mathsf{Q}XY}{\Gamma \vdash \mathsf{Q}YX}\ (\mathsf{Q}\mathrm{S}) \qquad \frac{\Gamma \vdash \mathsf{N}X \quad \Gamma \vdash \mathsf{Q}XY \quad \Gamma \vdash ZY}{\Gamma \vdash ZX}\ (\mathsf{Q}\mathrm{E}') \qquad \frac{\Gamma \vdash \mathsf{Q}\underline{0}(\mathsf{s}X)}{\Gamma \vdash Y}\ (\mathsf{Q}_\perp)$$

$$\frac{\Gamma \vdash \mathsf{N}X \quad \Gamma \vdash \mathsf{Q}XY}{\Gamma \vdash \mathsf{Q}(\mathsf{s}X)(\mathsf{s}Y)}\ (\mathsf{Qs}_+) \qquad \frac{\Gamma \vdash \mathsf{N}X \quad \Gamma \vdash \mathsf{Q}(\mathsf{s}X)(\mathsf{s}Y)}{\Gamma \vdash \mathsf{Q}XY}\ (\mathsf{Qs}_-)$$

$$\frac{\Gamma \vdash \mathsf{N}X \quad \Gamma, \mathsf{Q}X\underline{0} \vdash ZY_1 \quad \Gamma, \mathsf{N}x, \mathsf{Q}X(\mathsf{s}x) \vdash ZY_2 \quad x \notin \mathrm{FV}(\Gamma, X, Z, Y_1, Y_2)}{\Gamma \vdash Z(\mathsf{z}XY_1Y_2)}\ (\mathsf{Qz})$$

$$\frac{\Gamma \vdash \perp}{\Gamma \vdash Y}\ (\perp\mathrm{E}) \qquad \frac{}{\Gamma \vdash \mathsf{H}\perp}\ (\perp\mathsf{H}) \qquad \frac{}{\Gamma \vdash \top}\ (\top\mathrm{I}) \qquad \frac{}{\Gamma \vdash \mathsf{H}\top}\ (\top\mathsf{H})$$

$$\frac{\Gamma \vdash X\underline{0} \quad \Gamma, \mathsf{N}x \vdash X(\mathsf{s}x) \quad x \notin \mathrm{FV}(\Gamma, X)}{\Gamma \vdash \Xi\mathsf{N}X}\ (\mathsf{N}\mathrm{C})$$

$$\frac{\Gamma, \mathsf{N}x \vdash X(Yx) \quad x \notin \mathrm{FV}(\Gamma, X, Y)}{\Gamma \vdash \mathsf{FN}XY}\ (\mathsf{FN}\mathrm{I}) \qquad \frac{\Gamma \vdash \mathsf{FN}XY \quad \Gamma \vdash \mathsf{N}Z}{\Gamma \vdash X(YZ)}\ (\mathsf{FN}\mathrm{E})$$

$$\frac{\Gamma, X \vdash Z \quad X = Y}{\Gamma, Y \vdash Z}\ (\mathrm{EqL})$$

Figure 3.2: Admissible rules in illative systems containing arithmetic

Proof. The rule (QS) is derived as follows:

$$\dfrac{\Gamma \vdash \mathsf{Q}XY \qquad \dfrac{\dfrac{\Gamma \vdash \mathsf{N}X}{\Gamma \vdash \mathsf{Q}XX}\ (\mathsf{QI})}{\Gamma \vdash (\lambda x.\mathsf{Q}xX)X}\ (\text{Eq})}{\dfrac{\Gamma \vdash (\lambda x.\mathsf{Q}xX)Y}{\Gamma \vdash \mathsf{Q}YX}\ (\text{Eq})}\ (\mathsf{QE})$$

The rule (QE′) is derived using (QS) and (QE).

To derive the rule ($\mathsf{Q}_\bot$) assume $\Gamma \vdash \mathsf{Q}\underline{0}(\mathsf{s}X)$. Since $\mathsf{z}\underline{0}(\mathsf{N}\underline{0})Y = \mathsf{N}\underline{0}$ we have $\Gamma \vdash \mathsf{z}\underline{0}(\mathsf{N}\underline{0})Y$ by (NI_0) and (Eq). By (Eq) and (QE) we obtain $\Gamma \vdash \mathsf{z}(\mathsf{s}X)(\mathsf{N}\underline{0})Y$. But $\mathsf{z}(\mathsf{s}X)(\mathsf{N}\underline{0})Y = Y$, so $\Gamma \vdash Y$ by (Eq).

To derive (Qs_+) assume $\Gamma \vdash \mathsf{N}X$ and $\Gamma \vdash \mathsf{Q}XY$. We have $\Gamma \vdash \mathsf{Q}(\mathsf{s}X)(\mathsf{s}X)$ by (NI_s) and (QI). Then $\Gamma \vdash \mathsf{Q}(\mathsf{s}X)(\mathsf{s}Y)$ follows from (Eq) and (QE).

To derive (Qs_-) assume $\Gamma \vdash \mathsf{N}X$ and $\Gamma \vdash \mathsf{Q}(\mathsf{s}X)(\mathsf{s}Y)$. Then $\Gamma \vdash \mathsf{Q}(\mathsf{p}(\mathsf{s}X))(\mathsf{p}(\mathsf{s}X))$ by (QI) and (Eq), because $\mathsf{p}(\mathsf{s}X) = X$. Hence $\Gamma \vdash \mathsf{Q}XY$ follows from (Eq) and (QE).

The rule ($\bot$E) follows from ($\mathsf{Q}_\bot$). The rule ($\bot\mathsf{H}$) is derived using (NI_0), (NI_s) and (QH). The rule ($\top$I) follows from (NI_0) and (QI). The rule ($\top\mathsf{H}$) follows from (NI_0) and (QH). The rule (NC) is derived using (NInd) and (Weak). The rule (FNI) follows from (ΞNI) and (Eq). The rule (FNE) follows from (ΞNE) and (Eq).

To derive (Qz) assume $\Gamma \vdash \mathsf{N}X$, $\Gamma, \mathsf{Q}X\underline{0} \vdash ZY_1$ and $\Gamma, \mathsf{N}x, \mathsf{Q}X(\mathsf{s}x) \vdash ZY_2$, where $x \notin \mathrm{FV}(\Gamma, X, Z, Y_1, Y_2)$. We have $ZY_1 = Z(\mathsf{z}\,\underline{0}\,Y_1\,Y_2)$, so

$$\Gamma, \mathsf{Q}X\underline{0} \vdash Z(\mathsf{z}\,\underline{0}\,Y_1\,Y_2)$$

by (Eq). Since $\Gamma \vdash \mathsf{N}X$, using (Ax) and (QE′) we obtain

$$\Gamma, \mathsf{Q}X\underline{0} \vdash Z(\mathsf{z}XY_1Y_2)$$

Since $\Gamma \vdash \mathsf{N}X$, using ($\mathsf{NI}_0$) and (QH) we obtain $\Gamma \vdash \mathsf{H}(\mathsf{Q}X\underline{0})$. Therefore

$$(\star) \qquad \Gamma \vdash \mathsf{Q}X\underline{0} \supset Z(\mathsf{z}XY_1Y_2)$$

by (PI). On the other hand, we have $ZY_2 = Z(\mathsf{z}(\mathsf{s}x)Y_1Y_2)$, so

$$\Gamma, \mathsf{N}x, \mathsf{Q}X(\mathsf{s}x) \vdash Z(\mathsf{z}(\mathsf{s}x)Y_1Y_2)$$

by (Eq). Since $\Gamma \vdash \mathsf{N}X$, using (Ax) and (QE′) we obtain

$$\Gamma, \mathsf{N}x, \mathsf{Q}X(\mathsf{s}x) \vdash Z(\mathsf{z}XY_1Y_2).$$

Since $\Gamma, \mathsf{N}x \vdash \mathsf{N}X$ by (Weak), using (NI_s) and (QH) we obtain $\Gamma, \mathsf{N}x \vdash \mathsf{H}(\mathsf{Q}X(\mathsf{s}x))$. Therefore

$$(\star\star) \qquad \Gamma, \mathsf{N}x \vdash \mathsf{Q}X(\mathsf{s}x) \supset Z(\mathsf{z}XY_1Y_2)$$

Using ($\star$), ($\star\star$), (Eq) and (NC) we obtain

$$\Gamma \vdash \Xi\mathsf{N}(\lambda x.\mathsf{Q}Xx \supset Z(\mathsf{z}XY_1Y_2))$$

Hence, because $\Gamma \vdash \mathsf{N}X$, by ($\Xi$NE) and (Eq) we have

$$\Gamma \vdash \mathsf{Q}XX \supset Z(\mathsf{z}XY_1Y_2)$$

Since $\Gamma \vdash \mathsf{N}X$, we have $\Gamma \vdash \mathsf{Q}XX$ by (QI), so we finally obtain

$$\Gamma \vdash Z(\mathsf{z}XY_1Y_2)$$

by (PE).

The admissibility of (EqL) follows from (Weak), (Cut) and (Eq):

$$\dfrac{\dfrac{\dfrac{}{\Gamma, Y \vdash Y}\,(\mathrm{Ax}) \quad Y = X}{\Gamma, Y \vdash X}\,(\mathrm{Eq}) \quad \dfrac{\Gamma, X \vdash Z}{\Gamma, Y, X \vdash Z}\,(\mathrm{Weak})}{\Gamma, Y \vdash Z}\,(\mathrm{Cut})$$

□

Like in Section 1.1 we use the notation $\forall x : \mathsf{N} \,.\, X$ for $\Xi\mathsf{N}(\lambda x \,.\, X)$. We abbreviate $\Xi\mathsf{N}(\lambda x \,.\, \Xi\mathsf{N}(\lambda y \,.\, X))$ by $\forall x, y : \mathsf{N} \,.\, X$, etc. By $Y^k(X)$ we denote k-time application of Y to X, e.g., $Y^2(X) \equiv Y(YX)$. We assume $Y^0(X) \equiv X$.

Lemma 3.3.5. *We have* $\vdash \forall x : \mathsf{N} \,.\, \mathsf{Q}x(\mathsf{s}(x)) \supset Y$ *for any term* Y*.*

Proof. First note that $\vdash \mathsf{Q}\underline{0}(\mathsf{s}(\underline{0})) \supset Y$ follows from ($\mathsf{Q}_\bot$), (PI), (NI_0), (NI_s) and (QH). Thus by (NInd), (Eq), (PI), (NI_s) and (QH) it suffices to show

$$\mathsf{N}x, (\lambda y \,.\, \mathsf{Q}y(\mathsf{s}(y)) \supset Y)x, \mathsf{Q}(\mathsf{s}x)(\mathsf{s}(\mathsf{s}(x))) \vdash Y.$$

But this follows from (NI_s), (Qs_-), (Eq) and (PE). □

For the sake of brevity, from now on we shall only give sketches of formal proofs. We will use some rules implicitly, in particular, the rules (Weak), (Eq), (EqL), (NI_0), (NI_s), (FNE) and (ΞNE). When using the rule (NInd) we shall refer to the assumption Xx in the second premise as the *formal inductive hypothesis*, to the second premise as the *inductive step*, and to the first premise as the *base case*. When using the rule (Qz) we refer to the second premise as the *case for zero*, and to the third premise as the *case for successor*. Analogous terminology is used with (NC). For the sake of readability, we often write **ifz** X **then** Y **else** Z instead of $\mathsf{z}XYZ$.

Many of the following lemmas are not particularly surprising, because any illative system containing arithmetic essentially incorporates primitive recursive arithmetic (PRA). For some background on PRA see [TvD88, Chapter 3] and [Goo57, Cur41a]. We will not make this observation precise. Instead, we directly derive the requisite properties of a few terms representing certain recursive functions.

Definition 3.3.6. We set $\mathsf{flip} \equiv \lambda x.\mathsf{z}\, x\, \underline{1}\, \underline{0}$. We define the following terms by the recursive equations:

$$
\begin{aligned}
\mathsf{iter} &= \lambda f x y.\mathbf{ifz}\, x\, \mathbf{then}\, y\, \mathbf{else}\, (f(\mathsf{iter} f(\mathsf{p}x)y)) \\
\mathsf{even} &= \lambda x.\mathbf{ifz}\, x\, \mathbf{then}\, \underline{0}\, \mathbf{else}\, \mathsf{flip}(\mathsf{even}(\mathsf{p}x)) \\
\mathsf{div}_2 &= \lambda x.\mathbf{ifz}\, x\, \mathbf{then}\, \underline{0}\, \mathbf{else}\, \mathbf{ifz}\, \mathsf{p}z\, \mathbf{then}\, \underline{0}\, \mathbf{else}\, \mathsf{s}(\mathsf{div}_2(\mathsf{p}^2(x))) \\
\mathsf{inv}_1 &= \lambda x.\mathbf{ifz}\, \mathsf{even}\, x\, \mathbf{then}\, \mathsf{s}(\mathsf{inv}_1(\mathsf{div}_2\, x))\, \mathbf{else}\, \underline{0} \\
\mathsf{inv}_2 &= \lambda x.\mathbf{ifz}\, \mathsf{even}\, x\, \mathbf{then}\, \mathsf{inv}_2(\mathsf{div}_2\, x)\, \mathbf{else}\, \mathsf{div}_2(\mathsf{p}x) \\
\mathsf{eq} &= \lambda x y.\mathbf{ifz}\, x\, \mathbf{then}\, y\, \mathbf{else}\, \mathbf{ifz}\, y\, \mathbf{then}\, \underline{1}\, \mathbf{else}\, \mathsf{eq}(\mathsf{p}x)(\mathsf{p}y)
\end{aligned}
$$

We set

$$
\begin{aligned}
\mathsf{add}_2 &\equiv \lambda x.\mathsf{s}^2(x) \\
\mathsf{mul}_2 &\equiv \lambda x.\mathsf{iter}\, \mathsf{add}_2\, x\, \underline{0} \\
\mathsf{pow}_2 &\equiv \lambda x y.\mathsf{iter}\, \mathsf{mul}_2\, x\, y \\
\mathsf{m} &\equiv \lambda x y.\mathsf{pow}_2\, x\, (\mathsf{s}(\mathsf{mul}_2\, y)) \\
\mathsf{fst} &\equiv \lambda x.\mathbf{ifz}\, x\, \mathbf{then}\, \underline{0}\, \mathbf{else}\, \mathsf{inv}_1\, x \\
\mathsf{snd} &\equiv \lambda x.\mathbf{ifz}\, x\, \mathbf{then}\, \underline{0}\, \mathbf{else}\, \mathsf{inv}_2\, x
\end{aligned}
$$

For $k \in \mathbb{N}$ we also set

$$\mathsf{nth}_k \equiv \lambda x.\mathsf{snd}^k(\mathsf{fst}\, x)$$

We define Le by the recursive equation

$$\mathsf{Le} = \lambda x y.\mathbf{ifz}\, x\, \mathbf{then}\, \top\, \mathbf{else}\, \mathbf{ifz}\, y\, \mathbf{then}\, \bot\, \mathbf{else}\, \mathsf{Le}(\mathsf{p}x)(\mathsf{p}y)$$

The following lemma sheds some light on the meaning of the terms defined above. The terms we will ultimately need are m, fst, snd, nth_k and Le. Other terms are only needed to implement them. What we need is the pairing operation m which encodes pairs of numbers by a number. The terms fst and snd implement the first and second projections. The term nth_k implements the operation of taking the k-th element of a list of natural numbers encoded in a single natural by repeated use of m. The term Le implements the less-or-equal predicate on natural numbers.

Lemma 3.3.7. *For $n, m \in \mathbb{N}$ we have the following equalities:*

- $\mathsf{even}\, \underline{2n} = \underline{0}$,
- $\mathsf{even}\, \underline{2n+1} = \underline{1}$,
- $\mathsf{add}_2\, \underline{n} = \underline{n+2}$,
- $\mathsf{mul}_2\, \underline{n} = \underline{2n}$,
- $\mathsf{pow}_2\, \underline{n}\, \underline{m} = \underline{2^n m}$,
- $\mathsf{div}_2\, \underline{2n} = \underline{n}$,
- $\mathsf{m}\, \underline{n}\, \underline{m} = \underline{2^n(2m+1)}$,
- $\mathsf{fst}\, \underline{2^n(2m+1)} = \underline{n}$,
- $\mathsf{snd}\, \underline{2^n(2m+1)} = \underline{m}$,

- $\mathsf{eq}\,\underline{n}\,\underline{m} = \underline{0}$ *iff* $n = m$.

Proof. By induction. □

Lemma 3.3.8. *If* $n, m \in \mathbb{N}$ *and* $n \leq m$ *then* $\vdash \mathsf{Le}\,\underline{n}\,\underline{m}$.

Proof. By induction. □

Lemma 3.3.9. *If* $n \in \mathbb{N}$ *then* $\vdash \mathsf{N}\underline{n}$.

Proof. Use (NI_0) and then (NI_s) repeatedly n times. □

Most of the following lemmas are rather straightforward. They collectively show that the desired properties of the pairing operator m and the projections fst and snd may be formally proved in an illative system containing arithmetic. The proofs of most of these lemmas are as one would ordinarily do them. Essentially, once we have established the types of the terms under consideration, derivations in illative combinatory logic are much the same as in ordinary logic.

Lemma 3.3.10.

1. $\vdash \mathsf{FNN}(\mathsf{flip})$,
2. $\vdash \mathsf{FNN}(\mathsf{even})$,
3. $\vdash \mathsf{FNN}(\mathsf{add}_2)$,
4. $\vdash \mathsf{FN}(\mathsf{FNN})(\mathsf{eq})$,
5. $\vdash \mathsf{FN}(\mathsf{FNH})(\mathsf{Le})$,
6. $\vdash \mathsf{FN}(\mathsf{FN}(\mathsf{FNN}))(\mathsf{z})$.

Proof.

1. We use (NC). We have $\vdash \mathsf{N}(\mathsf{flip}\,\underline{0})$ because $\mathsf{flip}\,\underline{0} = \underline{1}$. Since $\mathsf{flip}(\mathsf{s}x) = \underline{0}$, we also have $\mathsf{N}x \vdash \mathsf{N}(\mathsf{flip}(\mathsf{s}x))$.
2. Use (NInd) and the previous point.
3. Use (NI_s) twice.
4. Use (NInd) and (NC).
5. Use (NInd) and (NC).
6. Use (NC) and ($\Xi\mathsf{NI}$).

□

Lemma 3.3.11. *If* $\vdash \mathsf{FNN}X$ *then* $\vdash \mathsf{FN}(\mathsf{FNN})(\mathsf{iter}\,X)$.

Proof. Using (NInd) and ($\Xi\mathsf{NI}$). □

Corollary 3.3.12.

1. $\vdash \mathsf{FNN}(\mathsf{mul}_2)$,
2. $\vdash \mathsf{FN}(\mathsf{FNN})(\mathsf{pow}_2)$,
3. $\vdash \mathsf{FN}(\mathsf{FNN})(\mathsf{m})$.

Lemma 3.3.13. $\vdash \forall x, y : \mathsf{N} \,.\, \mathsf{Q}(\mathsf{eq}xy)\underline{0} \supset \mathsf{Q}xy$.

Proof. We use (NInd). It suffices to prove the following two judgements.

- $\mathsf{N}y \vdash \mathsf{Q}(\mathsf{eq}\underline{0}y)\underline{0} \supset \mathsf{Q}\underline{0}y$. Using Lemma 3.3.10 and (QH) we obtain

$$\mathsf{N}y \vdash \mathsf{H}(\mathsf{Q}(\mathsf{eq}\underline{0}y)\underline{0})$$

so by (PI) it suffices to show

$$\mathsf{N}y, \mathsf{Q}(\mathsf{eq}\underline{0}y)\underline{0} \vdash \mathsf{Q}\underline{0}y$$

We have $\mathsf{eq}\underline{0}y = y$, so it suffices to show

$$\mathsf{N}y, \mathsf{Q}y\underline{0} \vdash \mathsf{Q}\underline{0}y$$

But this follows from (QS).

- $\Gamma \vdash \forall y : \mathsf{N} \,.\, \mathsf{Q}(\mathsf{eq}(\mathsf{s}x)y)\underline{0} \supset \mathsf{Q}(\mathsf{s}x)y$ with Γ equal to

$$\mathsf{N}x, \forall y : \mathsf{N} \,.\, \mathsf{Q}(\mathsf{eq}xy)\underline{0} \supset \mathsf{Q}xy$$

We use (NC). By Lemma 3.3.10 and (QH) and (PI), for the case for zero it suffices to show

$$(\star) \qquad \Gamma, \mathsf{Q}(\mathsf{eq}(\mathsf{s}x)\underline{0})\underline{0} \vdash \mathsf{Q}(\mathsf{s}x)\underline{0}$$

But $\mathsf{eq}(\mathsf{s}x)\underline{0} = \underline{1}$, so

$$\Gamma, \mathsf{Q}(\mathsf{eq}(\mathsf{s}x)\underline{0})\underline{0} \vdash \bot$$

by (Eq) and (QS). Using ($\bot$E) we obtain ($\star$).

Therefore, it remains to show

$$\Gamma, \mathsf{N}y \vdash \mathsf{Q}(\mathsf{eq}(\mathsf{s}x)(\mathsf{s}y))\underline{0} \supset \mathsf{Q}(\mathsf{s}x)(\mathsf{s}y)$$

By Lemma 3.3.10, (QH) and (PI), it suffices to prove

$$(\star\star) \qquad \Gamma, \mathsf{N}y, \mathsf{Q}(\mathsf{eq}(\mathsf{s}x)(\mathsf{s}y))\underline{0} \vdash \mathsf{Q}(\mathsf{s}x)(\mathsf{s}y)$$

We have $\mathsf{eq}(\mathsf{s}x)(\mathsf{s}y) = \mathsf{eq}xy$, so using the formal inductive hypothesis with (ΞNE), and then (Eq) and (PE), we obtain

$$\Gamma, \mathsf{N}y, \mathsf{Q}(\mathsf{eq}(\mathsf{s}x)(\mathsf{s}y))\underline{0} \vdash \mathsf{Q}xy$$

Thus ($\star\star$) follows from (Qs_+).

□

Lemma 3.3.14. $\vdash \forall x : \mathsf{N} \,.\, \mathsf{Q}\underline{0}(\mathsf{even}(\mathsf{mul}_2\, x))$.

Proof. We use (NInd). The base case $\vdash \mathsf{Q}\underline{0}(\mathsf{even}(\mathsf{mul}_2\, \underline{0}))$ follows from $\mathsf{even}(\mathsf{mul}_2\, \underline{0}) = \mathsf{even}(\underline{0}) = \underline{0}$, and rules (Eq) and (QI). Hence it suffices to show $\Gamma \vdash \mathsf{Q}\underline{0}(\mathsf{even}(\mathsf{mul}_2(\mathsf{s}x)))$ with appropriate Γ. Because $\mathsf{flip}(\mathsf{flip}(\underline{0})) = \underline{0}$ we have $\vdash \mathsf{Q}\underline{0}(\mathsf{flip}(\mathsf{flip}(\underline{0})))$. Since $\Gamma \vdash \mathsf{Q}\underline{0}(\mathsf{mul}_2\, x)$, and $\Gamma \vdash \mathsf{N}\underline{0}$, we obtain

$$\Gamma \vdash \mathsf{Q}\underline{0}(\mathsf{flip}(\mathsf{flip}(\mathsf{even}(\mathsf{mul}_2\, x))))$$

by (QE′). Because $\mathsf{mul}_2(\mathsf{s}x) = \mathsf{s}(\mathsf{s}(\mathsf{mul}_2\, x))$ we have

$$\mathsf{even}(\mathsf{mul}_2(\mathsf{s}x)) = \mathsf{flip}(\mathsf{flip}(\mathsf{even}(\mathsf{mul}_2\, x))).$$

Therefore

$$\Gamma \vdash \mathsf{Q}\underline{0}(\mathsf{even}(\mathsf{mul}_2(\mathsf{s}x)))$$

□

Corollary 3.3.15. $\vdash \forall x : \mathsf{N} \,.\, \mathsf{Q}\underline{1}(\mathsf{even}(\mathsf{s}(\mathsf{mul}_2\, x)))$.

Proof. Follows from Lemma 3.3.14, using $\mathsf{even}(\mathsf{s}(\mathsf{mul}_2\, x)) = \mathsf{flip}(\mathsf{even}(\mathsf{mul}_2\, x))$, $\Gamma \vdash \mathsf{Q}\underline{1}(\mathsf{flip}\, \underline{0})$, and the rule (QE). □

Lemma 3.3.16. $\vdash \forall x : \mathsf{N} \,.\, \mathsf{Q}x(\mathsf{div}_2(\mathsf{mul}_2\, x))$.

Proof. We use (NInd). The base case follows from (QI) and $\mathsf{div}_2(\mathsf{mul}_2\, \underline{0}) = \underline{0}$. For the inductive step it suffices to show

$$(\star) \qquad \Gamma \vdash \mathsf{Q}(\mathsf{s}x)(\mathsf{div}_2(\mathsf{mul}_2(\mathsf{s}x)))$$

with Γ equal to $\mathsf{N}x, \mathsf{Q}x(\mathsf{div}_2(\mathsf{mul}_2\, x))$. Since $\Gamma \vdash \mathsf{N}x$, by (Qs_+) we have

$$\Gamma \vdash \mathsf{Q}(\mathsf{s}x)(\mathsf{s}(\mathsf{div}_2(\mathsf{mul}_2\, x))).$$

We also have

$$\mathsf{div}_2(\mathsf{mul}_2(\mathsf{s}x)) = \mathsf{div}_2(\mathsf{s}^2(\mathsf{mul}_2\, x)) = \mathsf{s}(\mathsf{div}_2(\mathsf{mul}_2\, x))$$

so $(\star)$ follows by (Eq). □

Lemma 3.3.17. $\vdash \forall x, y : \mathsf{N} \,.\, \mathsf{Q}x(\mathsf{inv}_1(\mathsf{m}xy))$.

Proof. We use (NInd). It suffices to show the following two judgements.

- $\mathsf{N}y \vdash \mathsf{Q}\underline{0}(\mathsf{inv}_1(\mathsf{m}\,\underline{0}\,y))$. By Corollary 3.3.15 we have

$$(\star) \qquad \mathsf{N}y \vdash \mathsf{Q}\underline{1}(\mathsf{even}(\mathsf{s}(\mathsf{mul}_2\, y)))$$

  We have $\mathsf{N}y \vdash \mathsf{Q}\underline{0}\underline{0}$, so

$$\mathsf{N}y \vdash \mathsf{Q}\underline{0}(\mathsf{z}\underline{1}(\mathsf{s}(\mathsf{inv}_1(\mathsf{div}_2(\mathsf{m}\,\underline{0}\,y))))\underline{0})$$

by (Eq). Thus using (QE) with ($\star$) we obtain

$$\mathsf{N}y \vdash \mathsf{Q}\underline{0}(\mathsf{z}(\mathsf{even}(\mathsf{s}(\mathsf{mul}_2\, y)))(\mathsf{s}(\mathsf{inv}_1(\mathsf{div}_2(\mathsf{m}\,\underline{0}\,y))))\underline{0})$$

Because $\mathsf{m}\,\underline{0}\,y = \mathsf{s}(\mathsf{mul}_2\, y)$, we have

$$\mathsf{N}y \vdash \mathsf{Q}\underline{0}(\mathsf{inv}_1(\mathsf{m}\,\underline{0}\,y))$$

by (Eq).

- $\Gamma \vdash \mathsf{Q}(\mathsf{s}x)(\mathsf{inv}_1(\mathsf{m}(\mathsf{s}x)y))$ with Γ equal to

$$\mathsf{N}x, \mathsf{N}y, \forall y : \mathsf{N}\,.\,\mathsf{Q}x(\mathsf{inv}_1(\mathsf{m}xy))$$

From the formal inductive hypothesis we obtain

$$\Gamma \vdash \mathsf{Q}x(\mathsf{inv}_1(\mathsf{m}xy))$$

Since $\Gamma \vdash \mathsf{N}x$, by ($\mathsf{Qs}_+$) we have

$$\Gamma \vdash \mathsf{Q}(\mathsf{s}x)(\mathsf{s}(\mathsf{inv}_1(\mathsf{m}xy)))$$

Since $\Gamma \vdash \mathsf{H}(\mathsf{m}xy)$ by Corollary 3.3.12, using Lemma 3.3.16 and (QE) we obtain

$$\Gamma \vdash \mathsf{Q}(\mathsf{s}x)(\mathsf{s}(\mathsf{inv}_1(\mathsf{div}_2(\mathsf{mul}_2(\mathsf{m}xy)))))$$

Using (Eq) we get

$$\Gamma \vdash \mathsf{Q}(\mathsf{s}x)(\mathsf{z}\underline{0}(\mathsf{s}(\mathsf{inv}_1(\mathsf{div}_2(\mathsf{mul}_2(\mathsf{m}xy)))))\underline{0})$$

By Corollary 3.3.12 we have $\Gamma \vdash \mathsf{N}(\mathsf{m}xy)$. So using Lemma 3.3.14 and (QE) we obtain

$$\Gamma \vdash \mathsf{Q}(\mathsf{s}x)(\mathsf{z}(\mathsf{even}(\mathsf{mul}_2(\mathsf{m}xy)))(\mathsf{s}(\mathsf{inv}_1(\mathsf{div}_2(\mathsf{mul}_2(\mathsf{m}xy)))))\underline{0})$$

We have $\mathsf{m}(\mathsf{s}x)y = \mathsf{mul}_2(\mathsf{m}xy)$, so

$$\mathsf{inv}_1(\mathsf{m}(\mathsf{s}x)y) = \mathsf{z}(\mathsf{even}(\mathsf{mul}_2(\mathsf{m}xy)))(\mathsf{s}(\mathsf{inv}_1(\mathsf{div}_2(\mathsf{mul}_2(\mathsf{m}xy)))))\underline{0}$$

Therefore, by (Eq) we finally obtain

$$\Gamma \vdash \mathsf{Q}(\mathsf{s}x)(\mathsf{inv}_1(\mathsf{m}(\mathsf{s}x)y))$$

□

Lemma 3.3.18. $\vdash \forall x, y : \mathsf{N}\,.\,\mathsf{Q}y(\mathsf{inv}_2(\mathsf{m}xy))$.

Proof. We use (NInd). It suffices to show the following two judgements.

- $\mathsf{N}y \vdash \mathsf{Q}y(\mathsf{inv}_2(\mathsf{m}\,\underline{0}\,y))$. By Corollary 3.3.15 we have

$$(\star) \qquad \mathsf{N}y \vdash \mathsf{Q}\underline{1}(\mathsf{even}(\mathsf{s}(\mathsf{mul}_2\,y)))$$

  We have $\mathsf{N}y \vdash \mathsf{Q}yy$, so

$$\mathsf{N}y \vdash \mathsf{Q}y(\mathsf{z}\underline{1}(\mathsf{inv}_2(\mathsf{div}_2(\mathsf{m}\,\underline{0}\,y)))y)$$

  by (Eq). Thus using (QE) with $(\star)$ we obtain

$$\mathsf{N}y \vdash \mathsf{Q}y(\mathsf{z}(\mathsf{even}(\mathsf{s}(\mathsf{mul}_2\,y)))(\mathsf{inv}_2(\mathsf{div}_2(\mathsf{m}\,\underline{0}\,y)))y)$$

  Hence by Lemma 3.3.16 and (QE) we obtain

$$\mathsf{N}y \vdash \mathsf{Q}y(\mathsf{z}(\mathsf{even}(\mathsf{s}(\mathsf{mul}_2\,y)))(\mathsf{inv}_2(\mathsf{div}_2(\mathsf{m}\,\underline{0}\,y)))(\mathsf{div}_2(\mathsf{p}(\mathsf{s}(\mathsf{mul}_2\,y)))))$$

  so

$$\mathsf{N}y \vdash \mathsf{Q}y(\mathsf{z}(\mathsf{even}(\mathsf{s}(\mathsf{mul}_2\,y)))(\mathsf{inv}_2(\mathsf{div}_2(\mathsf{m}\,\underline{0}\,y)))(\mathsf{div}_2(\mathsf{p}(\mathsf{s}(\mathsf{mul}_2\,y)))))$$

  Because $\mathsf{m}\,\underline{0}\,y = \mathsf{s}(\mathsf{mul}_2\,y)$, we finally obtain

$$\mathsf{N}y \vdash \mathsf{Q}y(\mathsf{inv}_2(\mathsf{m}\,\underline{0}\,y))$$

  by (Eq).

- $\Gamma \vdash \mathsf{Q}y(\mathsf{inv}_2(\mathsf{m}(\mathsf{s}x)y))$ with Γ equal to

$$\mathsf{N}x, \mathsf{N}y, \forall y : \mathsf{N}\,.\,\mathsf{Q}y(\mathsf{inv}_2(\mathsf{m}xy))$$

  From the formal inductive hypothesis we obtain

$$\Gamma \vdash \mathsf{Q}y(\mathsf{inv}_2(\mathsf{m}xy))$$

  Since $\Gamma \vdash \mathsf{H}(\mathsf{m}xy)$ by Corollary 3.3.12, using Lemma 3.3.16 and (QE) we obtain

$$\Gamma \vdash \mathsf{Q}y(\mathsf{inv}_2(\mathsf{div}_2(\mathsf{mul}_2(\mathsf{m}xy))))$$

  Using (Eq) we get

$$\Gamma \vdash \mathsf{Q}y(\mathsf{z}\underline{0}(\mathsf{inv}_2(\mathsf{div}_2(\mathsf{mul}_2(\mathsf{m}xy))))(\mathsf{div}_2(\mathsf{p}(\mathsf{m}(\mathsf{s}x)y))))$$

  By Corollary 3.3.12 we have $\Gamma \vdash \mathsf{N}(\mathsf{m}xy)$. So using Lemma 3.3.14 and (QE) we obtain

$$\Gamma \vdash \mathsf{Q}y(\mathsf{z}(\mathsf{even}(\mathsf{mul}_2(\mathsf{m}xy)))(\mathsf{inv}_2(\mathsf{div}_2(\mathsf{mul}_2(\mathsf{m}xy))))(\mathsf{div}_2(\mathsf{p}(\mathsf{m}(\mathsf{s}x)y))))$$

  We have $\mathsf{m}(\mathsf{s}x)y = \mathsf{mul}_2(\mathsf{m}xy)$, so

$$\mathsf{inv}_2(\mathsf{m}(\mathsf{s}x)y) = \mathsf{z}(\mathsf{even}(\mathsf{mul}_2(\mathsf{m}xy)))(\mathsf{inv}_2(\mathsf{div}_2(\mathsf{mul}_2(\mathsf{m}xy))))(\mathsf{div}_2(\mathsf{p}(\mathsf{m}(\mathsf{s}x)y)))$$

  Therefore, by (Eq) we finally obtain

$$\Gamma \vdash \mathsf{Q}y(\mathsf{inv}_2(\mathsf{m}(\mathsf{s}x)y))$$

□

Lemma 3.3.19. $\vdash \forall x : \mathsf{N} \,.\, \mathsf{Q}\underline{0}(\mathsf{mul}_2\, x) \supset \mathsf{Q}\underline{0}x$.

Proof. We use (NC). The case for zero is obvious. By (PI) and Corollary 3.3.12, for the case for successor it suffices to show $\mathsf{N}x, \mathsf{Q}\underline{0}(\mathsf{mul}_2(\mathsf{s}x)) \vdash \mathsf{Q}\underline{0}$. We have $\mathsf{mul}_2(\mathsf{s}x) = \mathsf{s}^2(\mathsf{mul}_2)$, so this follows from $(\mathsf{Q}_\bot)$. □

Lemma 3.3.20. $\vdash \forall x, y : \mathsf{N} \,.\, \mathsf{Q}\underline{0}(\mathsf{pow}_2\, x\, y) \supset \mathsf{Q}\underline{0}y$.

Proof. We use (NInd). The base case is obvious, because $\mathsf{pow}_2\, \underline{0}\, y = y$. By (ΞNI), (PI) and Corollary 3.3.12, for the inductive step it suffices to show $\Gamma \vdash \mathsf{Q}\underline{0}y$ with Γ equal to

$$\mathsf{N}x, \mathsf{N}y, \mathsf{Q}\underline{0}(\mathsf{pow}_2(\mathsf{s}x)y), \forall z : \mathsf{N} \,.\, \mathsf{Q}\underline{0}(\mathsf{pow}_2\, x\, z) \supset \mathsf{Q}\underline{0}z$$

We have $\mathsf{pow}_2(\mathsf{s}x)y = \mathsf{mul}_2(\mathsf{pow}_2\, x\, y)$, so $\Gamma \vdash \mathsf{Q}\underline{0}(\mathsf{mul}_2(\mathsf{pow}_2\, x\, y))$. Using Corollary 3.3.12 and Lemma 3.3.19 we obtain $\Gamma \vdash \mathsf{Q}\underline{0}(\mathsf{pow}_2\, x\, y)$. Hence $\Gamma \vdash \mathsf{Q}\underline{0}y$ follows by the formal inductive hypothesis. □

Corollary 3.3.21. $\vdash \forall x, y : \mathsf{N} \,.\, \neg(\mathsf{Q}\underline{0}(\mathsf{m}xy))$.

Proof. Follows from Lemma 3.3.19, Lemma 3.3.20 and Corollary 3.3.12. □

Lemma 3.3.22.

1. $\vdash \forall x, y : \mathsf{N} \,.\, \mathsf{Q}x(\mathsf{fst}(\mathsf{m}xy))$,
2. $\vdash \forall x, y : \mathsf{N} \,.\, \mathsf{Q}y(\mathsf{snd}(\mathsf{m}xy))$.

Proof.

1. By (ΞNI) it suffices to show

$$\mathsf{N}x, \mathsf{N}y \vdash \mathsf{Q}x(\mathsf{fst}(\mathsf{m}xy)).$$

We have $\mathsf{fst}(\mathsf{m}xy) = \textbf{ifz}\, \mathsf{m}xy\, \textbf{then}\, \underline{0}\, \textbf{else}\, \mathsf{inv}_1(\mathsf{m}xy)$. By Corollary 3.3.12 we have

$$\mathsf{N}x, \mathsf{N}y \vdash \mathsf{N}(\mathsf{m}xy)$$

So we may use (Qz). For the case for zero we need to show

$$\mathsf{N}x, \mathsf{N}y, \mathsf{Q}(\mathsf{m}xy)\underline{0} \vdash \mathsf{Q}x\underline{0}.$$

This follows by Corollary 3.3.12, (QS), Corollary 3.3.21 and (⊥E). For the case for successor we need to show

$$\mathsf{N}x, \mathsf{N}y, \mathsf{N}z, \mathsf{Q}(\mathsf{m}xy)(\mathsf{s}z) \vdash \mathsf{Q}x(\mathsf{inv}_1(\mathsf{m}xy)).$$

This follows from Lemma 3.3.17.

2. Analogous to the previous point, using Lemma 3.3.18.

□

Lemma 3.3.23. $\vdash \forall x : \mathsf{N} \,.\, \mathsf{Le}\, x\, (\mathsf{s}x)$.

Proof. Using (NInd). □

Lemma 3.3.24. $\vdash \forall x, y, z : \mathsf{N} \,.\, \mathsf{Le}\, x\, y \supset \mathsf{Le}\, y\, z \supset \mathsf{Le}\, x\, z$.

Proof. We use (NInd). By Lemma 3.3.10, (PI) and ($\Xi\mathsf{NI}$), it suffices to show the following two judgements.

- $\mathsf{N}y, \mathsf{N}z, \mathsf{Le}\, \underline{0}\, y, \mathsf{Le}\, y\, z \vdash \mathsf{Le}\, \underline{0}\, z$. But $\mathsf{Le}\, \underline{0}\, z = \top$, so this holds.
- $\Gamma \vdash \forall y, z : \mathsf{N} \,.\, \mathsf{Le}(\mathsf{s}x)y \supset \mathsf{Le}\, y\, z \supset \mathsf{Le}(\mathsf{s}x)z$ where Γ is equal to

  $$\mathsf{N}x, \forall y, z : \mathsf{N} \,.\, \mathsf{Le}\, x\, y \supset \mathsf{Le}\, y\, z \supset \mathsf{Le}\, x\, z.$$

  We use (NC) twice. The cases for zero are easily shown using the definition of Le and the rule ($\bot$E). By Lemma 3.3.10 and (PI) it suffices to prove

  $$(\star) \qquad \Gamma' \vdash \mathsf{Le}(\mathsf{s}x)(\mathsf{s}z)$$

  where Γ' is equal to

  $$\Gamma, \mathsf{N}y, \mathsf{N}z, \mathsf{Le}(\mathsf{s}x)(\mathsf{s}y), \mathsf{Le}(\mathsf{s}y)(\mathsf{s}z).$$

  But we have $\Gamma' \vdash \mathsf{Le}\, x\, y$, because $\mathsf{Le}(\mathsf{s}x)(\mathsf{s}y) = \mathsf{Le}\, x\, y$. Similarly $\Gamma' \vdash \mathsf{Le}\, y\, z$. So using the formal inductive hypothesis we obtain

  $$\Gamma' \vdash \mathsf{Le}\, x\, z$$

  By (Eq) we conclude ($\star$).

□

Lemma 3.3.25. *The following rule is derivable.*

$$\frac{\Gamma \vdash X\underline{0} \quad \Gamma, \mathsf{N}x, \forall y : \mathsf{N} \,.\, \mathsf{Le}\, y\, x \supset Xy \vdash X(\mathsf{s}x) \quad x \notin \mathrm{FV}(\Gamma, X)}{\Gamma \vdash \Xi\mathsf{N}X} \ (\mathsf{N}\mathrm{Ind}')$$

Proof. Assume $\Gamma \vdash X\underline{0}$ and $\Gamma, \mathsf{N}y, \forall z : \mathsf{N} \,.\, \mathsf{Le}\, z\, y \supset Xz \vdash X(\mathsf{s}y)$ with $y \notin \mathrm{FV}(\Gamma, X)$. We show $\Gamma \vdash \forall x : \mathsf{N} \,.\, (\forall y : \mathsf{N} \,.\, \mathsf{Le}\, y\, x \supset Xy)$ using (NInd). From this $\Xi\mathsf{N}X$ follows using ($\Xi\mathsf{NI}$), twice ($\Xi\mathsf{N}$E), and (PE). So we need to prove the following.

- $\Gamma \vdash \forall y : \mathsf{N} \,.\, \mathsf{Le}\, y\, \underline{0} \supset Xy$. We use ($\mathsf{NC}$). The case for zero $\Gamma \vdash \mathsf{Le}\, \underline{0}\, \underline{0} \supset X\underline{0}$ follows from $\Gamma \vdash X\underline{0}$ and (PI). By Lemma 3.3.10 and (PI), for the case for successor it suffices to show

  $$\Gamma, \mathsf{N}y, \mathsf{Le}(\mathsf{s}y)\underline{0} \vdash X(\mathsf{s}y)$$

  But $\mathsf{Le}(\mathsf{s}y)\underline{0} = \bot$, so this follows from ($\bot$E).

- $\Gamma' \vdash \forall y : \mathsf{N} \,.\, \mathsf{Le}\, y\, (\mathsf{s}x) \supset Xy$ where Γ' is equal to

$$\Gamma, \mathsf{N}x, \forall y : \mathsf{N} \,.\, \mathsf{Le}\, y\, x \supset Xy$$

  We use (NC). The case for zero follows from $\Gamma \vdash X\underline{0}$ and (PI). By Lemma 3.3.10 and (PI), for the case for successor it suffices to show

$$\Gamma', \mathsf{N}y, \mathsf{Le}(\mathsf{s}y)(\mathsf{s}x) \vdash X(\mathsf{s}y).$$

  We will prove

$$(\star) \qquad \Gamma', \mathsf{N}y, \mathsf{Le}(\mathsf{s}y)(\mathsf{s}x) \vdash \forall z : \mathsf{N} \,.\, \mathsf{Le}\, z\, y \supset Xz$$

  By ($\Xi\mathsf{NI}$), Lemma 3.3.10 and (PI) it suffices to show

$$(\star\star) \qquad \Gamma', \mathsf{N}y, \mathsf{Le}(\mathsf{s}y)(\mathsf{s}x), \mathsf{N}z, \mathsf{Le}\, z\, y \vdash Xz$$

  But $\mathsf{Le}(\mathsf{s}y)(\mathsf{s}x) = \mathsf{Le}\, y\, x$, so by Lemma 3.3.24 we have

$$\Gamma', \mathsf{N}y, \mathsf{Le}(\mathsf{s}y)(\mathsf{s}x), \mathsf{N}z, \mathsf{Le}\, z\, y \vdash \mathsf{Le}\, z\, x$$

  Using the formal inductive hypothesis we obtain ($\star\star$). Hence ($\star$) holds.

  Now using the second assumption, i.e.,

$$\Gamma, \mathsf{N}y, \forall z : \mathsf{N} \,.\, \mathsf{Le}\, z\, y \supset Xz \vdash X(\mathsf{s}y)$$

  and ($\star$) with the rules (Weak) and (Cut) we obtain

$$\Gamma', \mathsf{N}y, \mathsf{Le}(\mathsf{s}y)(\mathsf{s}x) \vdash X(\mathsf{s}y)$$

  which is what we needed.

□

Like with (NInd), when using (NInd$'$) we also use the terminology of the base case, the inductive step and the formal inductive hypothesis.

Lemma 3.3.26. $\vdash \mathsf{FNN}(\mathsf{div}_2)$.

Proof. We use (NInd$'$). The base case follows from $\mathsf{div}_2\, \underline{0} = \underline{0}$. For the inductive step we need to show $\Gamma \vdash \mathsf{N}(\mathsf{div}_2(\mathsf{s}x))$ with appropriate Γ. We have $\mathsf{div}_2(\mathsf{s}x) = \mathsf{z}x\underline{0}(\mathsf{s}(\mathsf{div}_2(\mathsf{p}x)))$. We use ($\mathsf{Qz}$). The case for zero follows from $\vdash \mathsf{N}\underline{0}$. For the case for successor we need to show

$$(\star) \qquad \Gamma, \mathsf{N}y, \mathsf{Q}x(\mathsf{s}y) \vdash \mathsf{N}(\mathsf{s}(\mathsf{div}_2(\mathsf{p}x)))$$

We have $\Gamma, \mathsf{N}y \vdash \mathsf{Le}\, y\, (\mathsf{s}y)$ by Lemma 3.3.23. Therefore

$$\Gamma, \mathsf{N}y, \mathsf{Q}x(\mathsf{s}y) \vdash \mathsf{Le}\, y\, x$$

by (QE$'$). From the formal inductive hypothesis it now follows that

$$\Gamma, \mathsf{N}y, \mathsf{Q}x(\mathsf{s}y) \vdash \mathsf{N}(\mathsf{div}_2\, y)$$

so

$$\Gamma, \mathsf{N}y, \mathsf{Q}x(\mathsf{s}y) \vdash \mathsf{N}(\mathsf{div}_2(\mathsf{p}(\mathsf{s}y)))$$

Hence ($\star$) follows using (QE$'$) and (NI_s). □

Lemma 3.3.27. $\vdash \forall x : \mathsf{N} \,.\, \mathsf{Le}(\mathsf{div}_2\, x)x$.

Proof. Using (NInd′), (Qz), Lemma 3.3.23, Lemma 3.3.26 and Lemma 3.3.24. □

Lemma 3.3.28. $\vdash \forall x : \mathsf{N} \,.\, \mathsf{N}(\mathsf{inv}_1(\mathsf{s}x))$.

Proof. We use (NInd′). The base case is obvious, because $\mathsf{inv}_1(\mathsf{s}\underline{0}) = \underline{0}$. By ($\mathsf{NC}$) and ($\Xi$NE), for the inductive step we need to show $\Gamma \vdash \mathsf{N}(\mathsf{s}(\mathsf{inv}_1(\mathsf{div}_2(\mathsf{s}^2(x)))))$ with appropriate Γ. By Lemma 3.3.27 we have $\Gamma \vdash \mathsf{Le}(\mathsf{div}_2\, x)x$, and by Lemma 3.3.26 we have $\Gamma \vdash \mathsf{N}(\mathsf{div}_2\, x)$. So using the formal inductive hypothesis and (NI_s) we obtain $\Gamma \vdash \mathsf{N}(\mathsf{s}(\mathsf{inv}_1(\mathsf{s}(\mathsf{div}_2\, x))))$. But $\mathsf{div}_2(\mathsf{s}^2(x)) = \mathsf{s}(\mathsf{div}_2\, x)$, so we are done by (Eq). □

Lemma 3.3.29. $\vdash \forall x : \mathsf{N} \,.\, \mathsf{N}(\mathsf{inv}_2(\mathsf{s}x))$.

Proof. Similar to the proof of Lemma 3.3.28. We use (NInd′). The base case is obvious, because $\mathsf{inv}_2(\mathsf{s}\underline{0}) = \mathsf{div}_2\, \underline{0} = \underline{0}$. By ($\mathsf{NC}$) and ($\Xi$NE), for the inductive step we need to show $\Gamma \vdash \mathsf{N}(\mathsf{inv}_2(\mathsf{div}_2(\mathsf{s}^2(x))))$ with appropriate Γ. By Lemma 3.3.27 we have $\Gamma \vdash \mathsf{Le}(\mathsf{div}_2\, x)x$, and by Lemma 3.3.26 we have $\Gamma \vdash \mathsf{N}(\mathsf{div}_2\, x)$. So using the formal inductive hypothesis we obtain $\Gamma \vdash \mathsf{N}(\mathsf{inv}_2(\mathsf{s}(\mathsf{div}_2\, x)))$. But $\mathsf{div}_2(\mathsf{s}^2(x)) = \mathsf{s}(\mathsf{div}_2\, x)$, so we are done by (Eq). □

Corollary 3.3.30.

1. $\vdash \mathsf{FNN}(\mathsf{fst})$,
2. $\vdash \mathsf{FNN}(\mathsf{snd})$,
3. $\vdash \mathsf{FNN}(\mathsf{nth}_k)$.

Proof. Follows from (NC), Lemma 3.3.28 and Lemma 3.3.29. □

Lemma 3.3.31. $\vdash \forall x : \mathsf{N} \,.\, \mathsf{Le}(\mathsf{inv}_1(\mathsf{s}x))x$.

Proof. We use (NInd′). The base case follows by computation. For the inductive step we need to show $\Gamma \vdash \mathsf{Le}(\mathsf{inv}_1(\mathsf{s}^2(x)))(\mathsf{s}x)$ with appropriate Γ. Since

$$\begin{aligned}\mathsf{inv}_1(\mathsf{s}^2(x)) &= \textbf{ifz}\,\mathsf{even}(\mathsf{s}^2(x))\,\textbf{then}\,\mathsf{s}(\mathsf{inv}_1(\mathsf{div}_2(\mathsf{s}^2(x))))\,\textbf{else}\,\underline{0} \\ &= \textbf{ifz}\,\mathsf{even}(\mathsf{s}^2(x))\,\textbf{then}\,\mathsf{s}(\mathsf{inv}_1(\mathsf{s}(\mathsf{div}_2\, x)))\,\textbf{else}\,\underline{0}\end{aligned}$$

and $\Gamma \vdash \mathsf{N}(\mathsf{even}(\mathsf{s}^2(x)))$ by Lemma 3.3.10. Hence we may use (NC) and (ΞNE). Obviously, $\Gamma \vdash \mathsf{Le}\,\underline{0}\,(\mathsf{s}x)$, so it suffices to show

$$(\star) \qquad \Gamma \vdash \mathsf{Le}(\mathsf{s}(\mathsf{inv}_1(\mathsf{s}(\mathsf{div}_2\, x))))(\mathsf{s}x)$$

By Lemma 3.3.27 we have $\Gamma \vdash \mathsf{Le}(\mathsf{div}_2\, x)x$. Because $\Gamma \vdash \mathsf{N}(\mathsf{div}_2\, x)$ by Lemma 3.3.26, using the formal inductive hypothesis we obtain

$$\Gamma \vdash \mathsf{Le}(\mathsf{inv}_1(\mathsf{s}(\mathsf{div}_2\, x)))(\mathsf{div}_2\, x)$$

Because $\Gamma \vdash \mathsf{N}(\mathsf{inv}_1(\mathsf{s}(\mathsf{div}_2\, x)))$ by Lemma 3.3.28, using Lemma 3.3.24 we obtain

$$\Gamma \vdash \mathsf{Le}(\mathsf{inv}_1(\mathsf{s}(\mathsf{div}_2\, x)))x$$

Hence ($\star$) follows by (Qs_+). □

Lemma 3.3.32. $\vdash \forall x : \mathsf{N} \,.\, \mathsf{Le}(\mathsf{inv}_2(\mathsf{s}x))x$.

Proof. We use (NInd$'$). The base case follows by computation. For the inductive step we need to show $\Gamma \vdash \mathsf{Le}(\mathsf{inv}_2(\mathsf{s}^2(x)))(\mathsf{s}x)$ with appropriate Γ. Since

$$\begin{aligned} \mathsf{inv}_2(\mathsf{s}^2(x)) &= \mathbf{ifz}\,\mathsf{even}(\mathsf{s}^2(x))\,\mathbf{then}\,\mathsf{inv}_2(\mathsf{div}_2(\mathsf{s}^2(x)))\,\mathbf{else}\,\mathsf{div}_2(\mathsf{s}x) \\ &= \mathbf{ifz}\,\mathsf{even}(\mathsf{s}^2(x))\,\mathbf{then}\,\mathsf{inv}_2(\mathsf{s}(\mathsf{div}_2\,x))\,\mathbf{else}\,\mathsf{div}_2(\mathsf{s}x) \end{aligned}$$

and $\Gamma \vdash \mathsf{N}(\mathsf{even}(\mathsf{s}^2(x)))$ by Lemma 3.3.10, we may use (NC) and (ΞNE). For the case for zero we need to show

$$\Gamma \vdash \mathsf{Le}(\mathsf{div}_2(\mathsf{s}x))(\mathsf{s}x)$$

This follows from Lemma 3.3.27. For the case for successor it suffices to show

$$(\star) \qquad \Gamma \vdash \mathsf{Le}(\mathsf{inv}_2(\mathsf{s}(\mathsf{div}_2\,x)))(\mathsf{s}x)$$

By Lemma 3.3.27 we have $\Gamma \vdash \mathsf{Le}(\mathsf{div}_2\,x)x$. Because $\Gamma \vdash \mathsf{N}(\mathsf{div}_2\,x)$ by Lemma 3.3.26, using the formal inductive hypothesis we obtain

$$\Gamma \vdash \mathsf{Le}(\mathsf{inv}_2(\mathsf{s}(\mathsf{div}_2\,x)))(\mathsf{div}_2\,x)$$

Because $\Gamma \vdash \mathsf{N}(\mathsf{inv}_2(\mathsf{s}(\mathsf{div}_2\,x)))$ by Lemma 3.3.29, using Lemma 3.3.24 we obtain

$$\Gamma \vdash \mathsf{Le}(\mathsf{inv}_2(\mathsf{s}(\mathsf{div}_2\,x)))x$$

Hence ($\star$) follows by Lemma 3.3.23 and Lemma 3.3.24. $\square$

Corollary 3.3.33.

1. $\vdash \forall x : \mathsf{N} \,.\, \mathsf{Le}(\mathsf{fst}(\mathsf{s}x))x$,
2. $\vdash \forall x : \mathsf{N} \,.\, \mathsf{Le}(\mathsf{snd}(\mathsf{s}x))x$.

Corollary 3.3.34.

1. $\vdash \forall x : \mathsf{N} \,.\, \mathsf{Le}(\mathsf{fst}\,x)x$,
2. $\vdash \forall x : \mathsf{N} \,.\, \mathsf{Le}(\mathsf{snd}\,x)x$.

Proof. Follows from (NC), Corollary 3.3.33, Corollary 3.3.30 and Lemma 3.3.24. $\square$

Corollary 3.3.35. $\vdash \forall x : \mathsf{N} \,.\, \mathsf{Le}(\mathsf{nth}_k(\mathsf{s}x))x$.

Proof. Follows from Corollary 3.3.33, Corollary 3.3.34, Corollary 3.3.30 and Lemma 3.3.24. $\square$

Having developed enough formal machinery, we may proceed to the derivation of the paradox. For this purpose we need some additional assumptions on the illative system.

Definition 3.3.36. A *Hilbert-style illative system* is a pair $\langle \Sigma, \mathcal{R} \rangle$ where Σ is a finite set of constants and $\mathcal{R}$ is a finite set of *rules*. Each rule is a pair $\langle \mathcal{P}, Y \rangle$ where $\mathcal{P} \subseteq \mathbb{T}(\Sigma)$ is a finite set of *premises*, and $Y \in \mathbb{T}(\Sigma)$ is the *conclusion*. A term $X \in \mathbb{T}(\Sigma)$ is a *direct consequence* of terms $X_1, \ldots, X_k$ if there is a rule $\langle \{Y_1, \ldots, Y_k\}, Y \rangle \in \mathcal{R}$ and terms $Z_1, \ldots, Z_m$ such that $X \equiv Y[x_1/Z_1, \ldots, x_m/Z_m]$ and $X_i \equiv Y_i[x_1/Z_1, \ldots, x_m/Z_m]$ for $i = 1, \ldots, k$. A term $X \in \mathbb{T}(\Sigma)$ is *derivable from a set of assumptions* $\Gamma \subseteq \mathbb{T}(\Sigma)$, denoted $\Gamma \vdash X$, if there exists a finite sequence of terms $X_1, \ldots, X_n$ such that $X \equiv X_n$ and for every $i \leq n$ the term X_i is either a member of Γ or it is a direct consequence of some terms $X_{i_1}, \ldots, X_{i_k}$ where $1 \leq i_1, \ldots, i_k < i$. A term $X \in \mathbb{T}(\Sigma)$ is *derivable*, denoted $\vdash X$, if it is derivable from the empty set of assumptions.

Note that a Hilbert-style illative system uniquely determines a general illative system (see Definition 3.1). We often confuse Hilbert-style illative systems with their corresponding general illative systems. In particular, we say that a Hilbert-style illative system contains arithmetic if the corresponding general illative system does. Note that in every Hilbert-style illative system the axiom (Ax) and the rules (Weak) and (Cut) are admissible.

Theorem 3.3.37 (Kleene-Rosser paradox)**.** *Any Hilbert-style illative system $\mathcal{I}$ containing arithmetic is inconsistent, i.e., $\vdash_{\mathcal{I}} Y$ for an arbitrary term Y.*

Proof. Let $\mathcal{I} = \langle \Sigma, \mathcal{R} \rangle$ be a Hilbert-style illative system containing arithmetic. Without loss of generality we may assume that the system is based on combinatory logic with weak equality. For the sake of concreteness, assume $\Sigma = \{c_1, c_2, \ldots, c_n\}$ and

$$\mathcal{R} = \{\langle \{c_1 x, y(c_2 x z)\}, x(c_2 y) \rangle, \langle \emptyset, x(c_2(\mathsf{K}x)) \rangle, \ldots\}.$$

We shall only give definitions and proofs for the first two rules in $\mathcal{R}$ and the first two constants in Σ. It should be evident that the following arguments may be straightforwardly adapted to the general case.

First, we define the *code* $\varphi(X) \in \mathbb{N}$ of a term $X \in \mathbb{T}(\Sigma)$ inductively:

- $\varphi(\mathsf{K}) = 2^1$, $\varphi(\mathsf{S}) = 2^2$,
- $\varphi(c_i) = 2^{i+2}$ for $i = 1, \ldots, n$,
- $\varphi(XY) = 2^0(2(2^{\varphi(X)}(2\varphi(Y) + 1)) + 1) = 2^{\varphi(X)+1}(2\varphi(Y) + 1) + 1$.

We set $\ulcorner X \urcorner = \underline{\varphi(X)}$, i.e., $\ulcorner X \urcorner$ is the numeral representing the code of X. We say that $\ulcorner X \urcorner$ is the *numeral code* of X. We define the term app by $\mathsf{app} \equiv \lambda xy.\mathsf{m}\underline{0}(\mathsf{m}xy)$. We have $\mathsf{app}\,\ulcorner X \urcorner\,\ulcorner Y \urcorner = \ulcorner XY \urcorner$ for any terms X, Y, by Lemma 3.3.7. We define the evaluator T by the recursive equation:

$$
\begin{array}{rcl}
\mathsf{T} & = & \lambda x. \\
& & \quad \textbf{ifz}\ \mathsf{even}\ x\ \textbf{then} \\
& & \quad\quad (\textbf{ifz}\ \mathsf{eq}\ x\ \underline{2}\ \textbf{then}\ \mathsf{K} \\
& & \quad\quad \textbf{else}\ \textbf{ifz}\ \mathsf{eq}\ x\ \underline{4}\ \textbf{then}\ \mathsf{S} \\
& & \quad\quad \textbf{else}\ \textbf{ifz}\ \mathsf{eq}\ x\ \underline{8}\ \textbf{then}\ c_1 \\
& & \quad\quad \textbf{else}\ c_2) \\
& & \quad \textbf{else}\ (\mathsf{T}(\mathsf{fst}(\mathsf{snd}\,x)))(\mathsf{T}(\mathsf{snd}(\mathsf{snd}\,x)))
\end{array}
$$

Using Lemma 3.3.7, it follows by induction that $\mathsf{T}\lceil X \rceil = X$ for any term X. We define the term R_1 implementing the first rule of $\mathcal{R}$:

$$
\begin{array}{rcl}
\mathsf{R}_1 & \equiv & \lambda abxyz. \\
& & \quad \mathbf{ifz}\, \mathsf{eq}\, a\, (\mathsf{app} \lceil c_1 \rceil x)\, \mathbf{then} \\
& & \quad\quad (\mathbf{ifz}\, \mathsf{eq}\, b\, (\mathsf{app}\, y\, (\mathsf{app}(\mathsf{app}\lceil c_2 \rceil x) z))\, \mathbf{then} \\
& & \quad\quad\quad \mathsf{app}\, x\, (\mathsf{app}\lceil c_2 \rceil y) \\
& & \quad\quad \mathbf{else}\, \lceil \top \rceil) \\
& & \quad \mathbf{else}\, \lceil \top \rceil
\end{array}
$$

Similarly, we set $\mathsf{R}_2 \equiv \lambda x.\mathsf{app}\, x\, (\mathsf{app}\lceil c_2 \rceil (\mathsf{app}\lceil \mathsf{K} \rceil x))$. Now the term Θ enumerating the numeral codes of derivable terms is defined by the recursive equation:

$$
\begin{array}{rcl}
\Theta & = & \lambda x. \\
& & \quad \mathbf{ifz}\, x\, \mathbf{then} \\
& & \quad\quad \lceil \top \rceil \\
& & \quad \mathbf{else\, ifz}\, \mathsf{nth}_0\, x\, \mathbf{then} \\
& & \quad\quad \mathsf{R}_1(\Theta(\mathsf{nth}_1\, x))(\Theta(\mathsf{nth}_2\, x))(\mathsf{nth}_3\, x)(\mathsf{nth}_4\, x)(\mathsf{nth}_5\, x) \\
& & \quad \mathbf{else} \\
& & \quad\quad \mathsf{R}_2(\mathsf{nth}_1\, x)
\end{array}
$$

It follows by induction on the length of derivation that if $\vdash_{\mathcal{I}} X$ then there is $n \in \mathbb{N}$ with $\Theta\underline{n} = \lceil X \rceil$. Indeed, let $\psi(n,m) = 2^n(2m+1)$ for $n,m \in \mathbb{N}$. If $\vdash X$ is derived by the first rule from the premises $c_1 X_1$ and $X_2(c_2 X_1 X_3)$, then $\Theta\underline{m} = \lceil X \rceil$ for

$$m = \psi(0, \psi(n, \psi(k, \psi(\varphi(X_1), \psi(\varphi(X_2), \psi(\varphi(X_3), 0)))))),$$

where $n,k \in \mathbb{N}$ such that $\Theta\underline{n} = \lceil c_1 X_1 \rceil$ and $\Theta\underline{k} = \lceil X_2(c_2 X_1 X_3) \rceil$ are obtained from the inductive hypothesis. If $\vdash X$ is derived by the second rule, then $X \equiv X'(c_2(\mathsf{K}X'))$ and $\Theta\underline{m} = \lceil X \rceil$ for $m = \psi(1, \psi(\varphi(X'), 0))$.

The converse is also true, i.e., for any $n \in \mathbb{N}$ the term $\Theta\underline{n}$ is the numeral code of a derivable term. In (9) below we will show that this may be proved formally in $\mathcal{I}$.

The term Ω enumerating the numeral codes of terms representing numerical functions is defined by

$$
\begin{array}{rcl}
\Omega & \equiv & \lambda x. \\
& & \quad \mathbf{ifz}\, \mathsf{eq}(\Theta(\mathsf{fst}\, x))(\mathsf{app}\lceil \mathsf{FNN} \rceil (\mathsf{snd}\, x))\, \mathbf{then} \\
& & \quad\quad \mathsf{snd}\, x \\
& & \quad \mathbf{else} \\
& & \quad\quad \lceil \mathsf{I} \rceil
\end{array}
$$

Finally, we set $\mathcal{U} \equiv \lambda x.\mathsf{T}(\Omega x)$. It is clear that for every X such that $\vdash \mathsf{FNN}X$ there is $n \in \mathbb{N}$ with $\Omega\underline{n} = \lceil X \rceil$. Indeed, it suffices to take $n = 2^m(2\varphi(X)+1)$ where $m \in \mathbb{N}$ is such that $\Theta\underline{m} = \lceil \mathsf{FNN}X \rceil$.

To derive the paradox we shall prove the following conditions, and then apply Proposition 3.3.1.

(1) $\vdash \mathsf{FN}(\mathsf{FNN})(\mathsf{app})$.

(2) $\vdash \forall x_1, \ldots, x_5 : \mathsf{N} \,.\, \mathsf{N}(\mathsf{R}_1 x_1 \ldots x_5)$.

(3) $\vdash \mathsf{FNNR}_2$.

(4) $\vdash \mathsf{FNN}\Theta$.

(5) If $\Gamma \vdash \mathsf{N}X$, $\Gamma \vdash \mathsf{N}Y$ and $\Gamma \vdash Z(\mathsf{T}(\mathsf{app}\, X\, Y))$ then $\Gamma \vdash Z(\mathsf{T}X(\mathsf{T}Y))$.

(6) If $\Gamma \vdash \mathsf{N}X$, $\Gamma \vdash \mathsf{N}Y$ and $\Gamma \vdash Z(\mathsf{T}X(\mathsf{T}Y))$ then $\Gamma \vdash Z(\mathsf{T}(\mathsf{app}\, X\, Y))$.

(7) If $\Gamma \vdash \mathsf{T}X_1$, $\Gamma \vdash \mathsf{T}X_2$ and $\Gamma \vdash \mathsf{N}X_i$ for $i = 1, \ldots, 5$ then $\Gamma \vdash \mathsf{T}(\mathsf{R}_1 X_1 X_2 X_3 X_4 X_5)$.

(8) If $\Gamma \vdash \mathsf{N}X$ then $\Gamma \vdash \mathsf{T}(\mathsf{R}_2 X)$.

(9) $\vdash \forall x : \mathsf{N}.\mathsf{T}(\Theta x)$.

(10) $\vdash \forall x : \mathsf{N}.\mathsf{FNN}(\mathsf{T}(\Omega x))$.

We proceed with the proof of (1) – (10).

(1) Follows from Corollary 3.3.12.

(2) Follows from (1), Lemma 3.3.10 and Lemma 3.3.9.

(3) Follows from (1) and Lemma 3.3.9.

(4) We use (NInd′). The base case follows from $\vdash \mathsf{N}\ulcorner\top\urcorner$ (which holds by Lemma 3.3.9, because $\ulcorner\top\urcorner$ is a numeral). For the inductive step we need to show $\Gamma \vdash \mathsf{N}(\Theta(\mathsf{s}x))$ with appropriate Γ. We have

$$\begin{aligned}\Theta(\mathsf{s}x) \quad = \quad & \mathsf{z}(\mathsf{nth}_0(\mathsf{s}x)) \\ & (\mathsf{R}_1(\Theta(\mathsf{nth}_1(\mathsf{s}x)))(\Theta(\mathsf{nth}_2(\mathsf{s}x)))(\mathsf{nth}_3(\mathsf{s}x))(\mathsf{nth}_4(\mathsf{s}x))(\mathsf{nth}_5(\mathsf{s}x))) \\ & (\mathsf{R}_2(\mathsf{nth}_1(\mathsf{s}x)))\end{aligned}$$

By Corollary 3.3.30 we obtain

$$(\star_1) \qquad \Gamma \vdash \mathsf{N}(\mathsf{nth}_i(\mathsf{s}x))$$

for $i = 1, \ldots, 5$. So by (3) we have

$$(\star_2) \qquad \Gamma \vdash \mathsf{N}(\mathsf{R}_2(\mathsf{nth}_1(\mathsf{s}x)))$$

Using $(\star_1)$ and Corollary 3.3.35 we obtain $\Gamma \vdash \mathsf{Le}(\mathsf{nth}_i(\mathsf{s}x))x$ for $i = 1, 2$. Therefore

$$(\star_3) \qquad \Gamma \vdash \mathsf{N}(\Theta(\mathsf{nth}_i(\mathsf{s}x)))$$

for $i = 1, 2$ follows from the formal inductive hypothesis. Using $(\star_1)$, $(\star_3)$ and (2) we obtain

$$(\star_4) \qquad \Gamma \vdash \mathsf{N}(\mathsf{R}_1(\Theta(\mathsf{nth}_1(\mathsf{s}x)))(\Theta(\mathsf{nth}_2(\mathsf{s}x)))(\mathsf{nth}_3(\mathsf{s}x))(\mathsf{nth}_4(\mathsf{s}x))(\mathsf{nth}_5(\mathsf{s}x)))$$

Now $\Gamma \vdash \mathsf{N}(\Theta(\mathsf{s}x))$ follows from $(\star_1)$, $(\star_4)$, $(\star_2)$ and Lemma 3.3.10.

(5) Assume $\Gamma \vdash \mathsf{N}X$, $\Gamma \vdash \mathsf{N}Y$ and $\Gamma \vdash Z(\mathsf{T}(\mathsf{app}\, X\, Y))$. By (1) we have $\Gamma \vdash \mathsf{N}(\mathsf{app}\, X\, Y)$. We have $\mathsf{app}\, X\, Y = \mathsf{m}\underline{0}(\mathsf{m}XY) = \mathsf{s}(\mathsf{mul}_2(\mathsf{m}XY))$, so $\Gamma \vdash \mathsf{Q}\underline{1}(\mathsf{even}(\mathsf{app}\, X\, Y))$ by Corollary 3.3.15 and Corollary 3.3.12. Therefore, by (QE′) we obtain

$$\Gamma \vdash Z(\mathsf{T}(\mathsf{fst}(\mathsf{snd}(\mathsf{app}\, X\, Y)))(\mathsf{T}(\mathsf{snd}(\mathsf{snd}(\mathsf{app}\, X\, Y)))))$$

because

$$\begin{array}{rcl} \mathsf{T}(\mathsf{app}\, X\, Y) & = & \mathsf{z}\,(\mathsf{even}(\mathsf{app}\, X\, Y)) \\ & & (\mathsf{z}(\mathsf{eq}(\mathsf{app}\, X\, Y)\underline{2})\mathsf{K}(\mathsf{z}(\mathsf{eq}(\mathsf{app}\, X\, Y)\underline{4})\mathsf{S}(\mathsf{z}(\mathsf{eq}(\mathsf{app}\, X\, Y)\underline{8})c_1c_2))) \\ & & ((\mathsf{T}(\mathsf{fst}(\mathsf{snd}(\mathsf{app}\, X\, Y))))(\mathsf{T}(\mathsf{snd}(\mathsf{snd}(\mathsf{app}\, X\, Y))))) \end{array}$$

Because $\mathsf{app}\, X\, Y = \mathsf{m}\underline{0}(\mathsf{m}XY)$, $\Gamma \vdash X$, $\Gamma \vdash Y$ and $\Gamma \vdash \mathsf{N}(\mathsf{m}\, X\, Y)$ by Corollary 3.3.12, using Lemma 3.3.22 and (QE′) we obtain

$$\Gamma \vdash Z(\mathsf{T}X(\mathsf{T}Y)).$$

(6) Assume $\Gamma \vdash \mathsf{N}X$, $\Gamma \vdash \mathsf{N}Y$ and $\Gamma \vdash Z(\mathsf{T}X(\mathsf{T}Y))$. By Lemma 3.3.22 and (QE) we have $\Gamma \vdash Z(\mathsf{T}(\mathsf{fst}(\mathsf{m}XY))(\mathsf{T}(\mathsf{snd}(\mathsf{m}XY))))$. Because $\Gamma \vdash \mathsf{N}(\mathsf{m}XY)$ by Corollary 3.3.12, applying Lemma 3.3.22 and (QE) again, we obtain

$$\Gamma \vdash Z(\mathsf{T}(\mathsf{fst}(\mathsf{snd}(\mathsf{m}\underline{0}(\mathsf{m}XY))))(\mathsf{T}(\mathsf{snd}(\mathsf{snd}(\mathsf{m}\underline{0}(\mathsf{m}XY))))))$$

i.e.

$$\Gamma \vdash Z(\mathsf{T}(\mathsf{fst}(\mathsf{snd}(\mathsf{app}\, X\, Y)))(\mathsf{T}(\mathsf{snd}(\mathsf{snd}(\mathsf{app}\, X\, Y))))).$$

We have $\mathsf{app}\, X\, Y = \mathsf{m}\underline{0}(\mathsf{m}XY) = \mathsf{s}(\mathsf{mul}_2(\mathsf{m}XY))$, so $\Gamma \vdash \mathsf{Q}\underline{1}(\mathsf{even}(\mathsf{app}\, X\, Y))$ by Corollary 3.3.15 and Corollary 3.3.12. Therefore $\Gamma \vdash Z(\mathsf{T}(\mathsf{app}\, X\, Y))$ by (QE) and (Eq).

(7) Assume $\Gamma \vdash \mathsf{T}X_1$, $\Gamma \vdash \mathsf{T}X_2$ and $\Gamma \vdash \mathsf{N}X_i$ for $i = 1, \ldots, 5$. We have

$$\begin{array}{rcl} \mathsf{R}_1X_1 \ldots X_5 & = & \mathsf{z}\,(\mathsf{eq}\, X_1\,(\mathsf{app}\lceil c_1\rceil X_3)) \\ & & (\mathsf{z}\,(\mathsf{eq}\, X_2\,(\mathsf{app}\, X_4\,(\mathsf{app}(\mathsf{app}\lceil c_2\rceil X_3)X_5)))\,(\mathsf{app}\, X_3\,(\mathsf{app}\lceil c_2\rceil X_4))\,\lceil\top\rceil) \\ & & \lceil\top\rceil \end{array}$$

By (1), Lemma 3.3.9 and Lemma 3.3.10 we have

$$\Gamma \vdash \mathsf{N}(\mathsf{eq}\, X_1\,(\mathsf{app}\lceil c_1\rceil X_3)).$$

Hence we may use (Qz). The case for successor is obvious, because $\mathsf{T}\lceil\top\rceil = \top$. For the case for zero we need to show

$$\begin{array}{rcl} \Gamma, \mathsf{Q}(\mathsf{eq}\, X_1\,(\mathsf{app}\lceil c_1\rceil X_3))\underline{0} & \vdash & \mathsf{T}(\mathsf{z}\,(\mathsf{eq}\, X_2\,(\mathsf{app}\, X_4\,(\mathsf{app}(\mathsf{app}\lceil c_2\rceil X_3)X_5))) \\ & & \quad(\mathsf{app}\, X_3\,(\mathsf{app}\lceil c_2\rceil X_4)) \\ & & \quad\lceil\top\rceil) \end{array}$$

By (1), Lemma 3.3.9 and Lemma 3.3.10 we have

$$\Gamma \vdash \mathsf{N}(\mathsf{eq}\, X_2\,(\mathsf{app}\, X_4\,(\mathsf{app}(\mathsf{app}\lceil c_2\rceil X_3)X_5))).$$

Hence we may again use (Qz). The case for successor is again obvious. For the case for zero we need to show

$$(\star) \qquad \Gamma' \vdash \mathsf{T}(\mathsf{app}\, X_3\, (\mathsf{app}\lceil c_2 \rceil X_4))$$

with Γ' equal to

$$\Gamma, \mathsf{Q}(\mathsf{eq}\, X_1\, (\mathsf{app}\lceil c_1 \rceil X_3))\underline{0}, \mathsf{Q}(\mathsf{eq}\, X_2\, (\mathsf{app}\, X_4\, (\mathsf{app}(\mathsf{app}\lceil c_2 \rceil X_3) X_5)))\underline{0}.$$

Since $\Gamma' \vdash \mathsf{N}X_1$ and $\Gamma' \vdash \mathsf{N}(\mathsf{app}\lceil c_1 \rceil X_3)$ by Lemma 3.3.9 and (1), using Lemma 3.3.13 we obtain $\Gamma' \vdash \mathsf{Q}X_1(\mathsf{app}\lceil c_1 \rceil X_3)$. Since $\Gamma' \vdash \mathsf{T}X_1$, by ($\mathsf{Q}$E) we obtain $\Gamma' \vdash \mathsf{T}(\mathsf{app}\lceil c_1 \rceil X_3)$. Hence by (5) and $\mathsf{T}\lceil c_1 \rceil = c_1$ we have $\Gamma' \vdash c_1(\mathsf{T}X_3)$.

Because $\Gamma' \vdash \mathsf{N}X_2$ and $\Gamma' \vdash \mathsf{N}(\mathsf{app}\, X_4\, (\mathsf{app}(\mathsf{app}\lceil c_2 \rceil X_3) X_5))$ by Lemma 3.3.9 and (1), using Lemma 3.3.13 we obtain $\Gamma' \vdash \mathsf{Q}X_2(\mathsf{app}\, X_4\, (\mathsf{app}(\mathsf{app}\lceil c_2 \rceil X_3) X_5))$. Since $\Gamma' \vdash \mathsf{T}X_2$, by ($\mathsf{Q}$E) we obtain $\Gamma' \vdash \mathsf{T}(\mathsf{app}\, X_4\, (\mathsf{app}(\mathsf{app}\lceil c_2 \rceil X_3) X_5))$. Hence by (5), (1), Lemma 3.3.9 and $\mathsf{T}\lceil c_2 \rceil = c_2$ we have $\Gamma' \vdash \mathsf{T}X_4(c_2(\mathsf{T}X_3)(\mathsf{T}X_5))$.

Because $\Gamma' \vdash c_1(\mathsf{T}X_3)$ and $\Gamma' \vdash \mathsf{T}X_4(c_2(\mathsf{T}X_3)(\mathsf{T}X_5))$, by the first rule of $\mathcal{I}$ we obtain $\Gamma' \vdash \mathsf{T}X_3(c_2(\mathsf{T}X_4))$. By (6) this implies ($\star$).

(8) Follows using the second rule of $\mathcal{I}$ and (6).

(9) We use (NInd'). The base case is obvious, because $\mathsf{T}(\Theta\underline{0}) = \mathsf{T}\lceil\top\rceil = \top$. For the inductive step we need to show $\Gamma \vdash \mathsf{T}(\Theta(\mathsf{s}x))$ with appropriate Γ. We have

$$\begin{array}{rcl} \Theta(\mathsf{s}x) & = & \mathsf{z}(\mathsf{nth}_0(\mathsf{s}x)) \\ & & (\mathsf{R}_1(\Theta(\mathsf{nth}_1(\mathsf{s}x)))(\Theta(\mathsf{nth}_2(\mathsf{s}x)))(\mathsf{nth}_3(\mathsf{s}x))(\mathsf{nth}_4(\mathsf{s}x))(\mathsf{nth}_5(\mathsf{s}x))) \\ & & (\mathsf{R}_2(\mathsf{nth}_1(\mathsf{s}x))) \end{array}$$

By Corollary 3.3.30 we have $\Gamma \vdash \mathsf{N}(\mathsf{nth}_i(\mathsf{s}x))$ for $i = 0, \ldots, 5$. We use (NC) and ($\Xi\mathsf{NE}$) with $\mathsf{nth}_0(\mathsf{s}x)$. For the case for zero we need to show

$$(\star) \qquad \Gamma \vdash \mathsf{T}(\mathsf{R}_1(\Theta(\mathsf{nth}_1(\mathsf{s}x)))(\Theta(\mathsf{nth}_2(\mathsf{s}x)))(\mathsf{nth}_3(\mathsf{s}x))(\mathsf{nth}_4(\mathsf{s}x))(\mathsf{nth}_5(\mathsf{s}x)))$$

By Corollary 3.3.35 we have $\Gamma \vdash \mathsf{Le}(\mathsf{nth}_i(\mathsf{s}x))x$ for $i = 1, 2$. Hence by the formal inductive hypothesis we obtain $\Gamma \vdash \mathsf{T}(\Theta(\mathsf{nth}_i(\mathsf{s}x)))$ for $i = 1, 2$. By (4) we also have $\Gamma \vdash \mathsf{N}(\Theta(\mathsf{nth}_i(\mathsf{s}x)))$ for $i = 1, 2$. Therefore, by (7) we conclude ($\star$).

For the case for successor it suffices to show

$$\Gamma \vdash \mathsf{T}(\mathsf{R}_2(\mathsf{nth}_1(\mathsf{s}x))).$$

This follows from $\Gamma \vdash \mathsf{N}(\mathsf{nth}_1(\mathsf{s}x))$ and (8).

(10) We use ($\Xi\mathsf{NI}$). We have

$$\mathsf{FNN}(\mathsf{T}(\Omega x)) = \mathsf{FNN}(\mathsf{T}(\mathsf{z}(\mathsf{eq}(\Theta(\mathsf{fst}\, x))(\mathsf{app}\lceil\mathsf{FNN}\rceil(\mathsf{snd}\, x)))(\mathsf{snd}\, x)\lceil\mathsf{I}\rceil))$$

By Corollary 3.3.30, Lemma 3.3.9, Lemma 3.3.10, (1) and (4), we obtain

$$\mathsf{N}x \vdash \mathsf{N}(\mathsf{eq}(\Theta(\mathsf{fst}\,x))(\mathsf{app}\lceil\mathsf{FNN}\rceil(\mathsf{snd}\,x))).$$

Thus we may use (Qz). The case for successor is obvious, because $\mathsf{T}\lceil\mathsf{I}\rceil = \mathsf{I}$, and we have $\vdash \mathsf{FNNI}$. For the case for zero we need to show

$$(\star) \qquad \Gamma \vdash \mathsf{FNN}(\mathsf{T}(\mathsf{snd}\,x))$$

with Γ equal to

$$\mathsf{N}x, \mathsf{Q}(\mathsf{eq}(\Theta(\mathsf{fst}\,x))(\mathsf{app}\lceil\mathsf{FNN}\rceil(\mathsf{snd}\,x)))\underline{0}.$$

By Corollary 3.3.30 and (4) we have $\mathsf{N}x \vdash \mathsf{N}(\Theta(\mathsf{fst}\,x))$. By Lemma 3.3.9, Corollary 3.3.30 and (1) we have $\mathsf{N}x \vdash \mathsf{N}(\mathsf{app}\lceil\mathsf{FNN}\rceil(\mathsf{snd}\,x))$. Hence by Lemma 3.3.13 we have

$$\Gamma \vdash \mathsf{Q}(\Theta(\mathsf{fst}\,x))(\mathsf{app}\lceil\mathsf{FNN}\rceil(\mathsf{snd}\,x)).$$

Because $\mathsf{N}x \vdash \mathsf{N}(\mathsf{fst}\,x)$ by Corollary 3.3.30, using (9) we obtain $\Gamma \vdash \mathsf{T}(\Theta(\mathsf{fst}\,x))$. Hence by ($\mathsf{Q}$E) we have $\Gamma \vdash \mathsf{T}(\mathsf{app}\lceil\mathsf{FNN}\rceil(\mathsf{snd}\,x))$. Thus by Corollary 3.3.30, Lemma 3.3.9 and (5) we obtain $\Gamma \vdash \mathsf{T}\lceil\mathsf{FNN}\rceil(\mathsf{T}(\mathsf{snd}\,x))$. Then $(\star)$ follows from $\mathsf{T}\lceil\mathsf{FNN}\rceil = \mathsf{FNN}$.

To conclude that $\mathcal{I}$ is inconsistent, it remains to check $(a) - (g)$ in Proposition 3.3.1.

(a) Follows from (Ax).

(b) Follows from (FNI) and (Eq).

(c) Follows from (FNE).

(d) Follows from (FNI) and (NI_s).

(e) Follows from (QI) and Lemma 3.3.5.

(f) Recall that $\mathcal{U} \equiv \lambda x.\mathsf{T}(\Omega x)$. We have shown above (just after the definition of Ω) that if $\vdash \mathsf{FNN}X$ then there is $n \in \mathbb{N}$ with $\Omega\underline{n} = \lceil X\rceil$, so also $\mathcal{U}\underline{n} = \mathsf{T}\lceil X\rceil = X$. Since $\vdash \mathsf{N}\underline{n}$ for $n \in \mathbb{N}$, the condition (f) follows.

(g) Follows from (10) above.

□

Our formulation of the Kleene-Rosser paradox reveals an essential incompatibility between an unrestricted induction principle (NInd) and a Hilbert-style formulation of an illative system. Actually, the strongest of our systems $\mathcal{I}^+$ from Chapter 7 has (NInd) as a derived rule. It does not contain arithmetic in the sense of Definition 3.3.2, because it does not have the primitive Q with required properties. Nonetheless, we conjecture that the arguments of the present section could be adapted to show that every Hilbert-style illative system which contains all rules of $\mathcal{I}^+$ is inconsistent.

In fact, by modifying the model construction for $\mathcal{I}$K from Section 5.2.2 it would not be difficult to show consistency of a natural deduction illative system $\mathcal{I}$ containing arithmetic in the sense of Definition 3.3.2. Theorem 3.3.37 would then imply that any Hilbert-style illative system containing all rules of $\mathcal{I}$ must be inconsistent, i.e., $\mathcal{I}$ would have no Hilbert-style formalisation. This situation may seem strange, but upon closer consideration it is not really so surprising. The essential difference between natural deduction and Hilbert-style formulations of illative systems is that it may be impossible to faithfully represent the judgements of a natural deduction illative system in the system itself. In other words, there might not exist a function ψ from judgements to terms, such that $\Gamma \vdash X$ iff $\vdash \psi(\Gamma \vdash X)$. Note that representing $X_1, \ldots, X_n \vdash X$ by $X_1 \supset \ldots \supset X_n \supset X$ does not work if the implication introduction rule is restricted like in (PI). Because the rules of a natural deduction system operate on judgements with possibly non-empty contexts, in the definition of the enumerator Θ we would need to operate on codes of terms representing judgments. Therefore, it may be impossible to define an enumerator Θ for which $\forall x : \mathsf{N} \,.\, \mathsf{T}(\Theta x)$ would be provable in the system. With Hilbert-style systems this difficulty does not arise, because the rules of Hilbert-style systems essentially operate on judgements with empty contexts, and a judgement $\vdash X$ may be simply represented by X. When coupled with an unrestricted induction principle, this property of Hilbert-style illative systems allows them to "say" too much about themselves, leading to an inconsistency.

Chapter 4

Propositional logic

4.1 Illative systems

Definition 4.1.1. The system $\mathcal{I}\mathrm{Jp}$ of *intuitionistic propositional illative combinatory logic* comes in three variants: $\mathcal{I}\mathrm{Jp}_{\lambda\beta\eta}$, $\mathcal{I}\mathrm{Jp}_{\lambda\beta}$ and $\mathcal{I}\mathrm{Jp}_{\mathrm{CL}w}$. They differ in the underlying reduction systems. Let Σ be a set of constants containing at least the illative constants P, $\mathsf{\Lambda}$, V and $\bot$. For $\mathcal{I}\mathrm{Jp}_{\lambda\beta\eta}$ and $\mathcal{I}\mathrm{Jp}_{\lambda\beta}$ the set of terms is $\mathbb{T}_{\lambda}(\Sigma)$, for $\mathcal{I}\mathrm{Jp}_{\mathrm{CL}w}$ it is $\mathbb{T}_{\mathrm{CL}}(\Sigma)$. By $\mathcal{I}\mathrm{Jp}$ we denote any of the three variants. We will give definitions and proofs for $\mathcal{I}\mathrm{Jp}_{\lambda\beta\eta}$, and only indicate what (usually minor) changes are needed for other variants.

We adopt the abbreviations (compare Section 1.1):

- $\mathsf{H}X \equiv \mathsf{P}XX$,
- $X \supset Y \equiv \mathsf{P}XY$,
- $X \wedge Y \equiv \mathsf{\Lambda}XY$,
- $X \vee Y \equiv \mathsf{V}XY$,
- $\neg X \equiv X \supset \bot$.

Intuitively, $\mathsf{H}X$ means "X is a proposition". See also Section 1.1.

As in Section 1.1, the symbols X, Y, Z, etc., stand for terms, and Γ, Γ', etc., stand for sets of terms. The notation Γ, X abbreviates $\Gamma \cup \{X\}$.

A judgement in $\mathcal{I}\mathrm{Jp}$ has the form $\Gamma \vdash X$ where Γ is finite. The rules of $\mathcal{I}\mathrm{Jp}$ are given in Figure 4.1. For the variant $\mathcal{I}\mathrm{Jp}_{\lambda\beta\eta}$ the equality $=$ in rule (Eq) is the $\beta\eta$-equality, for $\mathcal{I}\mathrm{Jp}_{\lambda\beta\eta}$ it is β-equality, and for $\mathcal{I}\mathrm{Jp}_{\mathrm{CL}w}$ it is weak equality. For an infinite set of terms Γ we write $\Gamma \vdash X$ if there exists a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash X$ is derivable.

The system $\mathcal{I}\mathrm{Kp}$ of *classical propositional illative combinatory logic* is obtained by adding to $\mathcal{I}\mathrm{Jp}$ the rule of excluded middle:

$$\frac{\Gamma \vdash \mathsf{H}X}{\Gamma \vdash X \vee \neg X}\ (\mathrm{EM})$$

We write $\Gamma \vdash_{\mathcal{I}\mathrm{Jp}} X$ when $\Gamma \vdash X$ is derivable in $\mathcal{I}\mathrm{Jp}$, and analogously for $\Gamma \vdash_{\mathcal{I}\mathrm{Kp}} X$. The subscript is dropped when obvious from the context.

A set of terms Γ is *consistent* if $\Gamma \nvdash \bot$.

Note that it is not possible to consistently add to $\mathcal{I}\mathrm{Jp}$ the unrestricted axiom of excluded middle: $\Gamma \vdash X \vee \neg X$. With this axiom it is easy to derive $\Gamma \vdash \bot$ using rule ($\mathsf{\vee E}$) and a term X such that $X = \neg X$.

Lemma 4.1.2. *The following rules are admissible in* $\mathcal{I}\mathrm{Jp}$ *and* $\mathcal{I}\mathrm{Kp}$*:*

$$\frac{\Gamma \vdash X}{\Gamma, Y \vdash X}\ \text{(Weak)} \qquad\qquad \frac{\Gamma \vdash X}{\Gamma[x/Y] \vdash X[x/Y]}\ \text{(Sub)}$$

$$\frac{\Gamma, X \vdash Z \quad X = Y}{\Gamma, Y \vdash Z}\ \text{(EqL)} \qquad\qquad \frac{\Gamma \vdash X \quad \Gamma, X \vdash Y}{\Gamma \vdash Y}\ \text{(Cut)}$$

Proof. The admissibility of (Weak) and (Sub) follows by straightforward induction on the length of derivation. The rule (Cut) is derived thus:

$$\frac{\Gamma \vdash X \qquad \dfrac{\Gamma, X \vdash Y \quad \dfrac{\Gamma \vdash X}{\Gamma \vdash \mathsf{H}X}\ (\mathsf{HI})}{\Gamma \vdash X \supset Y}\ (\mathsf{PI}_l)}{\Gamma \vdash Y}\ (\mathsf{PE})$$

The admissibility of (EqL) now follows from (Weak), (Cut) and (Eq):

$$\frac{\dfrac{\dfrac{}{\Gamma, Y \vdash Y}\ \text{(Ax)} \quad Y = X}{\Gamma, Y \vdash X}\ \text{(Eq)} \qquad \dfrac{\Gamma, X \vdash Z}{\Gamma, Y, X \vdash Z}\ \text{(Weak)}}{\Gamma, Y \vdash Z}\ \text{(Cut)}$$

In fact, it would not be difficult to prove the admissibility of (Cut) and (EqL) directly by induction on the length of derivation. □

Informally, the illative system $\mathcal{I}\mathrm{Kp}$ may be interpreted in a kind of three-valued logic, in the sense explained below. The truth tables for propositional connectives are in Figure 4.2. The symbol T stands for true, F for false, and N for neither. The tables agree with the ones used by Bunder [Bun73a],[CHS72, §15C5].

The tables may be interpreted in the following manner (see also [CHS72, §15C5]). If X is true, then T is assigned to X. If X is false (in some sense), then F is assigned to X. Otherwise X has the value N. A judgement $\Gamma \vdash X$ means that if all elements of Γ are true, then so is X.

For instance, we give an informal justification for the rule (PI_l). According to our interpretation, the premises of the rule mean that:

1. if all elements of Γ and X are true, then Y is also true,
2. if all elements of Γ are true, then $\mathsf{H}X$ is also true.

$$\frac{}{\Gamma, X \vdash X}\ (\mathrm{Ax}) \qquad \frac{}{\Gamma \vdash \mathsf{H}\bot}\ (\bot\mathsf{HI})$$

$$\frac{\Gamma, X \vdash Y \quad \Gamma \vdash \mathsf{H}X}{\Gamma \vdash X \supset Y}\ (\mathsf{PI}_l) \qquad \frac{\Gamma \vdash X \quad \Gamma \vdash X \supset Y}{\Gamma \vdash Y}\ (\mathsf{PE})$$

$$\frac{\Gamma \vdash Y}{\Gamma \vdash X \supset Y}\ (\mathsf{PI}_r)$$

$$\frac{\Gamma \vdash \mathsf{H}(X \supset Y) \quad \Gamma, \mathsf{H}X \vdash Z \quad \Gamma, Y \vdash Z}{\Gamma \vdash Z}\ (\mathsf{PHE}_l)$$

$$\frac{\Gamma, X \vdash \mathsf{H}Y \quad \Gamma \vdash \mathsf{H}X}{\Gamma \vdash \mathsf{H}(X \supset Y)}\ (\mathsf{PHI})$$

$$\frac{\Gamma \vdash X \quad \Gamma \vdash \mathsf{H}(X \supset Y)}{\Gamma \vdash \mathsf{H}Y}\ (\mathsf{PHE}_r)$$

$$\frac{\Gamma \vdash X \quad \Gamma \vdash Y}{\Gamma \vdash X \wedge Y}\ (\wedge\mathsf{I}) \qquad \frac{\Gamma \vdash X \wedge Y}{\Gamma \vdash X}\ (\wedge\mathsf{E}_l) \quad \frac{\Gamma \vdash X \wedge Y}{\Gamma \vdash Y}\ (\wedge\mathsf{E}_r)$$

$$\frac{\Gamma \vdash \mathsf{H}X \quad \Gamma, X \vdash \mathsf{H}Y}{\Gamma \vdash \mathsf{H}(X \wedge Y)}\ (\wedge\mathsf{HI}_l) \qquad \frac{\Gamma \vdash X \quad \Gamma \vdash \mathsf{H}(X \wedge Y)}{\Gamma \vdash \mathsf{H}Y}\ (\wedge\mathsf{HE}_l)$$

$$\frac{\Gamma \vdash \mathsf{H}Y \quad \Gamma, Y \vdash \mathsf{H}X}{\Gamma \vdash \mathsf{H}(X \wedge Y)}\ (\wedge\mathsf{HI}_r) \qquad \frac{\Gamma \vdash Y \quad \Gamma \vdash \mathsf{H}(X \wedge Y)}{\Gamma \vdash \mathsf{H}X}\ (\wedge\mathsf{HE}_r)$$

$$\frac{\Gamma \vdash \mathsf{H}(X \wedge Y) \quad \Gamma, \mathsf{H}X \vdash Z \quad \Gamma, \mathsf{H}Y \vdash Z}{\Gamma \vdash Z}\ (\wedge\mathsf{HE})$$

$$\frac{\Gamma \vdash X}{\Gamma \vdash X \vee Y}\ (\vee\mathsf{I}_l) \quad \frac{\Gamma \vdash Y}{\Gamma \vdash X \vee Y}\ (\vee\mathsf{I}_r) \qquad \frac{\Gamma \vdash X \vee Y \quad \Gamma, X \vdash Z \quad \Gamma, Y \vdash Z}{\Gamma \vdash Z}\ (\vee\mathsf{E})$$

$$\frac{\Gamma \vdash \mathsf{H}X \quad \Gamma \vdash \mathsf{H}Y}{\Gamma \vdash \mathsf{H}(X \vee Y)}\ (\vee\mathsf{HI})$$

$$\frac{\Gamma \vdash \mathsf{H}(X \vee Y) \quad \Gamma, X \vdash Z \quad \Gamma, Y \vdash Z \quad \Gamma, \mathsf{H}X, \mathsf{H}Y \vdash Z}{\Gamma \vdash Z}\ (\vee\mathsf{HE})$$

$$\frac{\Gamma \vdash X}{\Gamma \vdash \mathsf{H}X}\ (\mathsf{HI}) \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash X}\ (\bot\mathrm{E})$$

$$\frac{\Gamma \vdash X \quad X = Y}{\Gamma \vdash Y}\ (\mathrm{Eq})$$

Figure 4.1: Rules of $\mathcal{I}$Jp

$\mathsf{P}XY$

$X \backslash Y$	T	F	N
T	T	F	N
F	T	T	T
N	T	N	N

$\mathsf{\Lambda}XY$

$X \backslash Y$	T	F	N
T	T	F	N
F	F	F	F
N	N	F	N

$\mathsf{V}XY$

$X \backslash Y$	T	F	N
T	T	T	T
F	T	F	N
N	T	N	N

X	$\neg X$
T	F
F	T
N	N

X	$\mathsf{H}X$
T	T
F	T
N	N

Figure 4.2: Truth tables for propositional connectives

Suppose all elements of Γ are true. By 2 above $\mathsf{H}X$ is also true, which by the truth table for H means that X is either true or false. If X is true, then by 1 also Y is true, and by the table for P we conclude that $\mathsf{P}XY$ is true. If X is false, then $\mathsf{P}XY$ is true by the truth table for P. By this informal argument we conclude that the rule (PI_l) is correct.

Essentially, the above informal semantics is formalised in the notion of an $\mathcal{I}$Kp-model in Definition 4.1.12. Note that the connectives are "lazy", e.g., $X \vee Y$ is true if X is, irrespective of the value of Y. So if X is true then $X \vee Y$ is a true proposition, even if Y does not represent a well-formed proposition at all. This interpretation of the meaning of logical connectives enables us to omit many premises which otherwise would be necessary in introduction rules. This agrees with our goal of minimising restrictions in inference rules.

Systems of illative combinatory logic usually do not include H-elimination rules ($*\mathsf{HE}$). They generally strive to minimise the number of rules and illative constants. Usually, also $\mathsf{\Lambda}$ and V are defined in terms of other illative primitives and not taken as constants. However, it is not clear, even in classical setting, how to define $\mathsf{\Lambda}$ and V from P so as to obtain the unrestricted introduction and elimination rules as in Figure 4.1. Note that standard definitions do not work, because the derived rules would then have some additional restrictions, i.e., additional premises. In any case, we are less concerned with minimising the number of rules and illative constants. Our interest in illative systems lies more in the fact that by including untyped lambda calculus (or combinatory logic) unrestricted recursion is incorporated directly into the logic.

As for the H-elimination rules, they may seem strange at first sight, but they can be informally justified by the truth tables in Figure 4.2. There are three main reasons for including these rules.

1. The symmetry between introduction and elimination rules in natural deduction is restored. Usually, illative systems include only H-introduction rules ($*\mathsf{HI}$), without corresponding elimination rules.
2. The system $\mathcal{I}$Jp with the rules of H-elimination is complete w.r.t. the Kripke semantics in Section 4.1.1, but a system without them is not complete.

3. Without these rules some of the rules in Lemma 4.1.3 are not admissible. The rules of Lemma 4.1.3 are useful in practice, e.g. in [Cza13c] (see also [Cza13d]) the rule ($\neg$HE) is indispensable in some derivations.

In Section 4.1.3 we provide an equivalent alternative formulation of $\mathcal{I}$Kp. In this formulation, all rules correspond directly to, and in a sense generalise, standard principles of classical propositional logic.

Lemma 4.1.3. *The following rules are admissible in* $\mathcal{I}$Jp.

$$\frac{\Gamma \vdash \mathsf{H}(\neg X)}{\Gamma \vdash \mathsf{H}X}\ (\neg\mathsf{HE}) \qquad \frac{\Gamma \vdash \mathsf{H}X}{\Gamma \vdash \mathsf{H}(\neg X)} \qquad \frac{\Gamma \vdash X}{\Gamma \vdash \neg\neg X}$$

$$\frac{\Gamma \vdash \neg X}{\Gamma \vdash X \supset Y} \qquad \frac{\Gamma \vdash \neg X}{\Gamma \vdash \neg(X \wedge Y)} \qquad \frac{\Gamma \vdash \neg Y}{\Gamma \vdash \neg(X \wedge Y)}$$

$$\frac{\Gamma \vdash \neg X \vee \neg Y}{\Gamma \vdash \neg(X \wedge Y)} \qquad \frac{\Gamma \vdash \neg X \wedge \neg Y}{\Gamma \vdash \neg(X \vee Y)} \qquad \frac{\Gamma \vdash \neg(X \vee Y)}{\Gamma \vdash \neg X \wedge \neg Y}$$

Moreover, in $\mathcal{I}$Kp *also the following rules are admissible.*

$$\frac{\Gamma \vdash \neg(X \wedge Y)}{\Gamma \vdash \neg X \vee \neg Y} \qquad \frac{\Gamma \vdash \neg\neg X}{\Gamma \vdash X}$$

Proof. Easy. □

4.1.1 Kripke semantics

In this section we define Kripke semantics for $\mathcal{I}$Jp.

Definition 4.1.4. A *propositional illative combinatory algebra* (PICA) is a tuple

$$\mathcal{C} = \langle C, \cdot, \mathsf{k}, \mathsf{s}, \mathsf{h}, \mathsf{p}, \wedge, \vee, \neg, \bot \rangle$$

where $\langle C, \cdot, \mathsf{k}, \mathsf{s} \rangle$ is a combinatory algebra and $\mathsf{h}, \mathsf{p}, \wedge, \vee, \neg, \bot \in C$, i.e., it is simply a combinatory algebra with distinguished elements $\mathsf{h}, \mathsf{p}, \wedge, \vee, \neg, \bot$. Given a PICA $\mathcal{C}$ we often confuse $\mathcal{C}$ with C.

A PICA is *extensional* if its associated combinatory algebra is extensional. A PICA is a *propositional illative λ-model* if its associated combinatory algebra is a λ-model.

Definition 4.1.5. A *Kripke* $\mathcal{I}\mathrm{Jp}_{\lambda\beta\eta}$*-model* (respectively $\mathcal{I}\mathrm{Jp}_{\lambda\beta}$-model or $\mathcal{I}\mathrm{Jp}_{\mathrm{CL}w}$-model) is a tuple $\mathcal{S} = \langle \mathcal{C}, I, S, \leq, \sigma_0, \sigma_1 \rangle$ where:

- $\mathcal{C}$ is an extensional propositional illative combinatory algebra (respectively a propositional illative λ-model, or a propositional illative combinatory algebra) satisfying $\mathsf{h} \cdot a = \mathsf{p} \cdot a \cdot a$ and $\neg \cdot a = \mathsf{p} \cdot a \cdot \bot$ for any $a \in \mathcal{C}$,
- I is a function from Σ to $\mathcal{C}$ providing an interpretation for constants,

- S is a non-empty set of states,
- $\leq$ is a partial order on S,
- σ_0 and σ_1 are functions from $\mathcal{C}$ to $\mathbb{P}(S)$, satisfying the following for any $a, b \in \mathcal{C}$, where $\sigma_h(a) = \sigma_0(a) \cup \sigma_1(a)$:
  1. $\sigma_h(a)$ and $\sigma_1(a)$ are upward-closed[1] wrt. $\leq$,
  2. $\sigma_0(\bot) = S$,
  3. $\sigma_0(a) \cap \sigma_1(a) = \emptyset$,
  4. $\sigma_1(\mathsf{v} \cdot a \cdot b) = \sigma_1(a) \cup \sigma_1(b)$,
  5. $\sigma_0(\mathsf{v} \cdot a \cdot b) = \sigma_0(a) \cap \sigma_0(b)$,
  6. $\sigma_1(\wedge \cdot a \cdot b) = \sigma_1(a) \cap \sigma_1(b)$,
  7. $s \in \sigma_0(\wedge \cdot a \cdot b)$ iff
     - $s \in \sigma_0(a)$ and for every $s' \geq s$ such that $s' \in \sigma_1(a)$ we have $s' \in \sigma_h(b)$, or
     - $s \in \sigma_0(b)$ and for every $s' \geq s$ such that $s' \in \sigma_1(b)$ we have $s' \in \sigma_h(a)$,
  8. $s \in \sigma_1(\mathsf{p} \cdot a \cdot b)$ iff
     - $s \in \sigma_h(a)$ and for every $s' \geq s$ such that $s' \in \sigma_1(a)$ we have $s' \in \sigma_1(b)$, or
     - $s \in \sigma_1(b)$,
  9. $s \in \sigma_0(\mathsf{p} \cdot a \cdot b)$ iff
     - $s \in \sigma_h(a)$, and
     - for every $s' \geq s$ such that $s' \in \sigma_1(a)$ we have $s' \in \sigma_h(b)$, and
     - there exists $s' \geq s$ such that $s' \in \sigma_1(a)$ and $s' \in \sigma_0(b)$.

An $\mathcal{S}$-*valuation* is a function from V to $\mathcal{C}$ (cf. Definition 2.3.17). Given an $\mathcal{S}$-valuation $\rho : V \to \mathcal{C}$ we define the *value* of $M \in \mathbb{T}_{\mathrm{CL}}$, denoted $[\![M]\!]^{\mathcal{S}}_\rho$ or just $[\![M]\!]_\rho$, by induction on the structure of M:

- $[\![x]\!]_\rho = \rho(x)$ if $x \in V$,
- $[\![\mathsf{K}]\!]_\rho = \mathsf{k}$, $[\![\mathsf{S}]\!]_\rho = \mathsf{s}$,
- $[\![\mathsf{P}]\!]_\rho = \mathsf{p}$, $[\![\mathsf{V}]\!]_\rho = \mathsf{v}$, $[\![\mathsf{\Lambda}]\!]_\rho = \wedge$, $[\![\bot]\!]_\rho = \bot$,
- $[\![c]\!]_\rho = I(c)$ if $c \in \Sigma \setminus \{\mathsf{P}, \mathsf{V}, \mathsf{\Lambda}, \bot\}$,
- $[\![M_1 M_2]\!]_\rho = [\![M_1]\!]_\rho \cdot [\![M_1]\!]_\rho$.

For $M \in \mathbb{T}_\lambda$ we set $[\![M]\!]_\rho = [\![(M)_{\mathrm{CL}}]\!]_\rho$. We drop the subscript and/or the superscript when clear or irrelevant.

If $s \in \sigma_i([\![M]\!]_\rho)$, we write $s, \rho \Vdash_i M$. If M is closed then we use the notation $s \Vdash_i M$. We write $\mathcal{S}, \rho \Vdash_i M$ if $s, \rho \Vdash_i M$ for all $s \in \mathcal{S}$. We use the notation $s, \rho \Vdash_i \Gamma$ (resp. $\mathcal{S}, \rho \Vdash_i \Gamma$) if $s, \rho \Vdash_i M$ (resp. $\mathcal{S}, \rho \Vdash_i M$) for all $M \in \Gamma$. Finally, we write $\Gamma \Vdash_i M$ if for every $\mathcal{S}$, every $s \in \mathcal{S}$ and every ρ, the condition $s, \rho \Vdash_1 \Gamma$ implies $s, \rho \Vdash_i M$. Instead of $\Vdash_1$ we sometimes use $\Vdash$. To make it clear what kind of Kripke models are used we also write $\Gamma \Vdash_{\mathcal{I}\mathrm{Jp}} M$ for $\Gamma \Vdash_1 M$.

[1]A set $A \subseteq S$ is upward-closed wrt. $\leq$ iff $s \in A$ and $s' \geq s$ imply $s' \in A$.

Intuitively, $s \in \sigma_1(a)$ means that a is known to be a true proposition in state s, and $s \in \sigma_0(a)$ means that in state s, the element a is known to be a proposition which is not (known/forced to be) true. So $s \in \sigma_h(a) = \sigma_0(a) \cup \sigma_1(a)$ means that a is known to be a proposition in state s. Thus, if $s \in \sigma_0(a)$ then we may have $s' \in \sigma_1(a)$ for some $s' \geq s$. A proposition which is not true may become true with expanding our knowledge. However, if $s \in \sigma_0(a)$ then $s' \in \sigma_0(a) \cup \sigma_1(a)$ for all $s' \geq s$, because knowledge is monotonous – once we know a is a proposition it will be a proposition in any future state of knowledge. If a is a proposition which is not true, then in any future state, it may either remain so, or become true. That a is false in state s is expressed by $s \in \sigma_1(\mathsf{p} \cdot a \cdot \bot)$, i.e., that its negation is true, not by $s \in \sigma_0(a)$. A proposition is false in state s if it is a proposition which is not true in all states $s' \geq s$. If $s \in \sigma_h(a)$, i.e., a is a proposition in state s, then a is "always ultimately knowable", i.e., however we expand our knowlege, it is always possible to expand it further so that a becomes either true or false.

With regard to condition 7, its interpretation is as follows: $\wedge \cdot a \cdot b$ is a proposition which is not true in state s iff a is a proposition which is not true in state s or b is a proposition which is not true in state s, *and* $\wedge \cdot a \cdot b$ remains a proposition when we expand our knowledge. Condition 7 is formulated in a way to ensure that if $\wedge \cdot a \cdot b$ is a proposition which is not true in some state s then it remains a proposition in all states $s' \geq s$. If we took $\sigma_0(\wedge \cdot a \cdot b) = \sigma_0(a) \cup \sigma_0(b)$ then this might not be so. Similar considerations apply to the formulation of condition 9 for p.

Given a Kripke model $\mathcal{S}$, we often confuse $\mathcal{S}$ with S and we implicitly assume that k, s, p, etc., belong to the combinatory algebra associated with $\mathcal{S}$.

We use the notion of a *Kripke* $\mathcal{I}$Jp-*model* to refer generically to a Kripke $\mathcal{I}\mathrm{Jp}_{\lambda\beta\eta}$-, $\mathcal{I}\mathrm{Jp}_{\lambda\beta}$-, or $\mathcal{I}\mathrm{Jp}_{\mathrm{CL}w}$-model, when it does not matter exactly which one it is.

Note that the conditions on σ_1 and σ_0 above are not a definition of σ_1 or σ_0, but just some properties we wish σ_1 and σ_0 to satisfy. Because of the combinatory completeness of $\mathcal{C}$, it is not obvious that there exists a structure satisfying the above requirements.

Lemma 4.1.6. *In any Kripke* $\mathcal{I}$Jp-*model the following conditions hold for any* $s \in S$ *and* $a, b \in \mathcal{C}$*:*

- $s \in \sigma_1(\mathsf{h} \cdot a)$ *iff* $s \in \sigma_h(a)$*,*
- $s \in \sigma_1(\mathsf{p} \cdot a \cdot \bot)$ *iff for every* $s' \geq s$ *we have* $s' \in \sigma_0(a)$*,*
- $s \in \sigma_0(\mathsf{p} \cdot a \cdot \bot)$ *iff* $s \in \sigma_h(a)$ *and there exists* $s' \geq s$ *such that* $s' \in \sigma_1(a)$*,*
- *if* $s \in \sigma_1(\mathsf{h} \cdot a)$ *then for every* $s' \geq s$ *there exists* $s'' \geq s'$ *such that* $s'' \in \sigma_1(a)$ *or* $s'' \in \sigma_1(\mathsf{p} \cdot a \cdot \bot)$*.*

Proof. Follows easily from definitions. □

For convenience of reference, we now reformulate in terms of $\Vdash_1$ and $\Vdash_0$ some of the conditions on σ_1 and σ_0 in a Kripke $\mathcal{I}$Jp-model.

Lemma 4.1.7. *For any Kripke* $\mathcal{I}$Jp-*model* $\mathcal{S}$ *and any valuation* ρ *the following hold for* $s \in \mathcal{S}$ *and* $X, Y \in \mathbb{T}$*:*

1. $s, \rho \Vdash_1 X \vee Y$ *iff* $s, \rho \Vdash_1 X$ *or* $s, \rho \Vdash_1 Y$,
2. $s, \rho \Vdash_0 X \vee Y$ *iff* $s, \rho \Vdash_0 X$ *and* $s, \rho \Vdash_0 Y$,
3. $s, \rho \Vdash_1 X \wedge Y$ *iff* $s, \rho \Vdash_1 X$ *and* $s, \rho \Vdash_1 Y$,
4. $s, \rho \Vdash_0 X \wedge Y$ *iff*
    - $s, \rho \Vdash_0 X$ *and for every* $s' \geq s$ *such that* $s', \rho \Vdash_1 X$ *we have* $s', \rho \Vdash_1 \mathsf{H}Y$*, or*
    - $s, \rho \Vdash_0 Y$ *and for every* $s' \geq s$ *such that* $s', \rho \Vdash_1 Y$ *we have* $s', \rho \Vdash_1 \mathsf{H}X$,
5. $s, \rho \Vdash_1 X \supset Y$ *iff*
    - $s, \rho \Vdash_1 \mathsf{H}X$ *and for every* $s' \geq s$ *such that* $s', \rho \Vdash_1 X$ *we have* $s', \rho \Vdash_1 Y$*, or*
    - $s, \rho \Vdash_1 Y$,
6. $s, \rho \Vdash_0 X \supset Y$ *iff*
    - $s, \rho \Vdash_1 \mathsf{H}X$*, and*
    - *for every* $s' \geq s$ *such that* $s', \rho \Vdash_1 X$ *we have* $s', \rho \Vdash_1 \mathsf{H}Y$*, and*
    - *there exists* $s' \geq s$ *such that* $s', \rho \Vdash_1 X$ *and* $s', \rho \Vdash_0 Y$,
7. $s, \rho \nVdash_1 \bot$ *and* $s, \rho \Vdash_0 \bot$,
8. $s, \rho \Vdash_1 \mathsf{H}X$ *iff* $s, \rho \Vdash_1 X$ *or* $s, \rho \Vdash_0 X$.

Proof. Follows easily from definitions. □

Theorem 4.1.8 (Soundness of Kripke semantics for $\mathcal{I}$Jp)**.**
If $\Gamma \vdash_{\mathcal{I}\mathrm{Jp}} M$ *then* $\Gamma \Vdash_{\mathcal{I}\mathrm{Jp}} M$.

Proof. The proof is by fairly straightforward induction on the length of derivation of $\Gamma \vdash M$. Despite its easiness, we give the proof in full for the sake of completeness.

Assume $\mathcal{S}$ is a Kripke $\mathcal{I}$Jp-model, ρ a valuation and $s \in \mathcal{S}$. Suppose $s, \rho \Vdash_1 \Gamma$, and consider the last rule used in the derivation of $\Gamma \vdash M$. We show $s, \rho \Vdash_1 M$. The claim is obvious for the axioms (Ax) and ($\bot$HI).

(PI_l) Then $M \equiv X \supset Y$ and $\Gamma, X \vdash Y$ and $\Gamma \vdash \mathsf{H}X$. So $s, \rho \Vdash_1 \mathsf{H}X$ by the IH. Let $s' \geq s$ be such that $s', \rho \Vdash_1 X$. Then $s', \rho \Vdash_1 \Gamma, X$, and thus $s', \rho \Vdash_1 Y$ by the IH, because $\Gamma, X \vdash Y$. Therefore, $s, \rho \Vdash_1 X \supset Y$.

(PI_r) Then $M \equiv X \supset Y$ and $\Gamma \vdash Y$. By the IH we have $s, \rho \Vdash_1 Y$, so $s, \rho_1 \Vdash_1 M$.

(PE) Then $\Gamma \vdash N$ and $\Gamma \vdash N \supset M$. By the IH we have $s, \rho \Vdash_1 N$ and $s, \rho \Vdash_1 N \supset M$. This implies that $s, \rho \Vdash_1 M$.

(PHI) Then $M \equiv \mathsf{H}(X \supset Y)$ and $\Gamma, X \vdash \mathsf{H}Y$ and $\Gamma \vdash \mathsf{H}X$. By the IH, $s, \rho \Vdash_1 \mathsf{H}X$. Because $\Gamma, X \vdash \mathsf{H}Y$, by the IH for every $s' \geq s$ such that $s', \rho \Vdash_1 X$ we have $s', \rho \Vdash_1 Y$ or $s', \rho \Vdash_0 Y$, i.e., $s', \rho \Vdash_1 \mathsf{H}Y$. If there exists $s' \geq s$ such that $s', \rho \Vdash_1 X$ and $s', \rho \Vdash_0 Y$, then $s, \rho \Vdash_0 X \supset Y$. Otherwise, $s, \rho \Vdash_1 X \supset Y$. In any case, $s, \rho \Vdash_1 \mathsf{H}(X \supset Y)$.

(PHE_l) Then $\Gamma \vdash \mathsf{H}(X \supset Y)$, $\Gamma, \mathsf{H}X \vdash M$ and $\Gamma, Y \vdash M$. By the IH, $s, \rho \Vdash_1 X \supset Y$ or $s, \rho \Vdash_0 X \supset Y$. Suppose $s, \rho \Vdash_1 X \supset Y$. If $s, \rho \Vdash_1 Y$ then by the IH $s, \rho \Vdash_1 M$. Otherwise, $s, \rho \Vdash_1 \mathsf{H}X$, so $s, \rho \Vdash_1 M$ by the IH. Suppose $s, \rho \Vdash_0 X \supset Y$. Then $s, \rho \Vdash_1 \mathsf{H}X$, so $s, \rho \Vdash_1 M$ by the IH.

(PHE_r) Then $M \equiv \mathsf{H}Y$ and $\Gamma \vdash X$ and $\Gamma \vdash \mathsf{H}(X \supset Y)$. By the IH, $s, \rho \Vdash_1 X$, and either $s, \rho \Vdash_1 X \supset Y$ or $s, \rho \Vdash_0 X \supset Y$. If $s, \rho \Vdash_1 X \supset Y$ then $s, \rho \Vdash_1 Y$, hence $s, \rho \Vdash_1 \mathsf{H}Y$. If $s, \rho \Vdash_0 X \supset Y$ then also $s, \rho \Vdash_1 \mathsf{H}Y$, because $s, \rho \Vdash_1 X$.

($\mathsf{\Lambda I}$) Then $M \equiv X \wedge Y$ and $\Gamma \vdash X$ and $\Gamma \vdash Y$. By the IH, $s, \rho \Vdash_1 X$ and $s, \rho \Vdash_1 Y$. Thus $s, \rho \Vdash_1 X \wedge Y$.

($\mathsf{\Lambda}\mathrm{E}_l$) Then $\Gamma \vdash M \wedge N$. By the IH, $s, \rho \Vdash_1 M \wedge N$. Thus $s, \rho \Vdash_1 M$.

($\mathsf{\Lambda}\mathrm{E}_r$) Analogous to ($\mathsf{\Lambda}\mathrm{E}_l$).

($\mathsf{\Lambda HI}_l$) Then $M \equiv \mathsf{H}(X \wedge Y)$ and $\Gamma \vdash \mathsf{H}X$ and $\Gamma, X \vdash \mathsf{H}Y$. By the IH, $s, \rho \Vdash_1 X$ or $s, \rho \Vdash_0 X$, and for every $s' \geq s$ such that $s', \rho \Vdash_1 X$ we have $s', \rho \Vdash_1 \mathsf{H}Y$. If $s, \rho \Vdash_1 X$ then $s, \rho \Vdash_1 \mathsf{H}Y$. Thus $s, \rho \Vdash_1 Y$ or $s, \rho \Vdash_0 Y$. In the first case, $s, \rho \Vdash_1 X \wedge Y$, so $s, \rho \Vdash_1 \mathsf{H}(X \wedge Y)$. In the second case, or when $s, \rho \Vdash_0 X$, we have $s, \rho \Vdash_0 X \wedge Y$, so also $s, \rho \Vdash_1 \mathsf{H}(X \wedge Y)$.

($\mathsf{\Lambda HI}_r$) Analogous to ($\mathsf{\Lambda HI}_l$).

($\mathsf{\Lambda HE}$) Then $\Gamma \vdash \mathsf{H}(X \wedge Y)$, $\Gamma, \mathsf{H}X \vdash M$ and $\Gamma, \mathsf{H}Y \vdash M$. By the IH, $s, \rho \Vdash_1 X \wedge Y$ or $s, \rho \Vdash_0 X \wedge Y$. In any case, it is easy to check that $s, \rho \Vdash_1 \mathsf{H}X$ or $s, \rho \Vdash_1 \mathsf{H}Y$, and thus $s, \rho \Vdash_1 M$ by the IH.

($\mathsf{\Lambda HE}_l$) Then $M \equiv \mathsf{H}Y$ and $\Gamma \vdash X$ and $\Gamma \vdash \mathsf{H}(X \wedge Y)$. By the IH, $s, \rho \Vdash_1 X$, and either $s, \rho \Vdash_1 X \wedge Y$ or $s, \rho \Vdash_0 X \wedge Y$. If $s, \rho \Vdash_1 X \wedge Y$ then $s, \rho \Vdash_1 Y$. If $s, \rho \Vdash_0 X \wedge Y$ then $s, \rho \Vdash_0 X$ or $s, \rho \Vdash_0 Y$. Thus $s, \rho \Vdash_0 Y$ because $s, \rho \Vdash_1 X$. In any case, $s, \rho \Vdash_1 \mathsf{H}Y$.

($\mathsf{\Lambda HE}_r$) Analogous to ($\mathsf{\Lambda HE}_l$).

(VI_l) Then $M \equiv X \vee Y$ and $\Gamma \vdash X$. By the IH, $s, \rho \Vdash_1 X$. Thus $s, \rho \Vdash_1 X \vee Y$.

(VI_r) Analogous to (VI_l).

(VE) Then $\Gamma \vdash X \vee Y$ and $\Gamma, X \vdash M$ and $\Gamma, Y \vdash M$. By the IH, $s, \rho \Vdash_1 X$ or $s, \rho \Vdash_1 Y$. If $s, \rho \Vdash_1 X$ then $s, \rho \Vdash_1 \Gamma, X$, and thus $s, \rho \Vdash_1 M$ by the IH. If $s, \rho \Vdash_1 Y$ the proof is analogous.

(VHI) Then $M \equiv \mathsf{H}(X \vee Y)$ and $\Gamma \vdash \mathsf{H}X$ and $\Gamma \vdash \mathsf{H}Y$. By the IH, $s, \rho \Vdash_1 X$ or $s, \rho \Vdash_0 X$, and $s, \rho \Vdash_1 Y$ or $s, \rho \Vdash_0 Y$. It is easy to check that in any case $s, \rho \Vdash_1 \mathsf{H}(X \vee Y)$.

(VHE) Then $\Gamma \vdash \mathsf{H}(X \vee Y)$, $\Gamma, X \vdash M$, $\Gamma, Y \vdash M$ and $\Gamma, \mathsf{H}X, \mathsf{H}Y \vdash M$. By the IH, either $s, \rho \Vdash_1 X \vee Y$ or $s, \rho \Vdash_0 X \vee Y$. If $s, \rho \Vdash_1 X \vee Y$ then $s, \rho \Vdash_1 X$ or $s, \rho \Vdash_1 Y$, and thus $s, \rho \Vdash_1 M$ by the IH. If $s, \rho \Vdash_0 X \vee Y$ then $s, \rho \Vdash_0 X$ and $s, \rho \Vdash_0 Y$. Thus $s, \rho \Vdash_1 \mathsf{H}X$ and $s, \rho \Vdash_1 \mathsf{H}Y$, so $s, \rho \Vdash_1 M$ by the IH.

(HI) Then $M \equiv \mathsf{H}X$ and $\Gamma \vdash X$. By the IH, $s, \rho \Vdash_1 X$, so $s, \rho \Vdash_1 \mathsf{H}X$.

($\bot\mathrm{E}$) Then $\Gamma \vdash \bot$, so by the IH, $s, \rho \Vdash_1 \bot$. This is, however, impossible. Hence, there are no s and ρ such that $s, \rho \Vdash_1 \Gamma$, and the claim holds.

(Eq) Follows from the fact that $\mathcal{C}$ is an extensional combinatory algebra (for $\mathcal{I}\mathrm{Jp}_{\lambda\beta}$ a λ-model, for $\mathcal{I}\mathrm{Jp}_{\mathrm{CL}w}$ a combinatory algebra).

□

Our next aim is to prove completeness of Kripke semantics for $\mathcal{I}$Jp. For this purpose we need some auxiliary definitions and lemmas.

Definition 4.1.9. A set of terms Γ is *prime* if:

- it is closed under consequence in $\mathcal{I}$Jp, i.e., $\Gamma \vdash X$ implies $X \in \Gamma$,
- $\Gamma \vdash X \vee Y$ implies $\Gamma \vdash X$ or $\Gamma \vdash Y$.

Lemma 4.1.10. *For every Γ with $\Gamma \nvdash M$, there exists a prime $\Gamma' \supseteq \Gamma$ with $\Gamma' \nvdash M$.*

Proof. Consider the set $\mathcal{X} = \{\Gamma' \supseteq \Gamma \mid \Gamma' \nvdash M\}$ ordered by inclusion. Is is easy to see that every chain C of elements of $\mathcal{X}$ has an upper bound $\bigcup C \in \mathcal{X}$. Indeed, if $\bigcup C \vdash M$ then there exists a finite $\Gamma_0 \subseteq \bigcup C$ such that $\Gamma_0 \vdash M$. But since C is a chain and Γ_0 is finite, it must be a subset of some $\Gamma_1 \in C$. So $\Gamma_1 \vdash M$. Contradiction. Of course, also $\mathcal{X} \neq \emptyset$, because $\Gamma \in \mathcal{X}$.

Therefore, by the Kuratowski-Zorn Lemma, there exists a maximal element $\Gamma' \in \mathcal{X}$. To show that Γ' is prime it suffices to check:

- $\Gamma', X \nvdash M$ for any X such that $\Gamma' \vdash X$,
- if $\Gamma' \vdash X \vee Y$ then $\Gamma', X \nvdash M$ or $\Gamma', Y \nvdash M$.

For the first part, suppose $\Gamma' \vdash X$ and $\Gamma', X \vdash M$. Then $\Gamma' \vdash M$ by the derived rule (Cut). Contradiction. For the second part, assume $\Gamma' \vdash X \vee Y$ and $\Gamma', X \vdash M$ and $\Gamma', Y \vdash M$. Then $\Gamma' \vdash M$ by rule ($\vee$E). Contradiction. □

Theorem 4.1.11 (Completeness of Kripke semantics for $\mathcal{I}$Jp)**.**
If $\Gamma \Vdash_{\mathcal{I}\mathrm{Jp}} M$ then $\Gamma \vdash_{\mathcal{I}\mathrm{Jp}} M$.

Proof. Assume $\Gamma \nvdash M$. We construct a Kripke $\mathcal{I}$Jp-model $\mathcal{S} = \langle \mathcal{C}, I, S, \leq, \sigma_0, \sigma_1 \rangle$ and a valuation ρ such that there exists a state $s \in \mathcal{S}$ with $s, \rho \Vdash \Gamma$ but $s, \rho \nVdash M$.

As the carrier of $\mathcal{C}$ we take $\beta\eta$-equality (for $\mathcal{I}\mathrm{Jp}_{\lambda\beta}$: β-equality, for $\mathcal{I}\mathrm{Jp}_{\mathrm{CL}w}$: w-equality) equivalence classes of terms from $\mathbb{T}$. We will denote by $[X]$ the equivalence class of X. We use the notation $[\Gamma'] = \{[X] \mid X \in \Gamma'\}$. We take $\mathsf{k} = [\mathsf{K}]$, $\mathsf{s} = [\mathsf{S}]$, $\mathsf{p} = [\mathsf{P}]$, etc. As h we take $[\lambda x.\mathsf{P}xx]$ and as $\neg$ we take $[\lambda x.\mathsf{P}x\bot]$. Application is defined by $[X] \cdot [Y] = [XY]$. It is easy to check that $\mathcal{C}$ is a PICA which is extensional (for $\mathcal{I}\mathrm{Jp}_{\lambda\beta}$: a λ-model) and it satisfies $\mathsf{h} \cdot a = \mathsf{p} \cdot a \cdot a$ and $\neg \cdot a = \mathsf{p} \cdot a \cdot \bot$ for any $a \in \mathcal{C}$. We define the interpretation of constants I by $I(c) = [c]$.

The set of states S is defined as the set of all $[\Gamma']$ such that Γ' is a prime and consistent set of terms. Because $\Gamma \nvdash M$, the set S is non-empty, by Lemma 4.1.10. We define:

- $\sigma_1([X]) = \{[\Gamma'] \in S \mid \Gamma' \vdash X\}$,
- $\sigma_0([X]) = \{[\Gamma'] \in S \mid \Gamma' \vdash \mathsf{H}X \text{ and } \Gamma' \nvdash X\}$.

So states are sets of equivalence classes. As the order on states we take set inclusion.

Note that σ_1 and σ_0 are well-defined because of the presence of rule (Eq). Note also that:

- $[\Gamma'] \in \sigma_1([X])$ iff $\Gamma' \vdash X$,
- $[\Gamma'] \in \sigma_0([X])$ iff $\Gamma' \not\vdash X$ and $\Gamma' \vdash \mathsf{H}X$,
- $[\Gamma'] \in \sigma_h([X])$ iff $\Gamma' \vdash \mathsf{H}X$.

This follows from the fact that Γ' is closed under consequence, from the rule (Eq) and the derived rule (EqL). Therefore, we may identify states with prime and consistent sets of terms, indentify $\sigma_1([X])$ with the set of prime and consistent Γ' such that $\Gamma' \vdash X$, and analogously for σ_0 and σ_h.

It remains to check that the conditions on σ_0 and σ_1 from Definition 4.1.5 are satisfied.

1. It is obvious that $\sigma_1([X])$ and $\sigma_h([X])$ are upward-closed, because the ordering is by inclusion.
2. Follows from the rule ($\bot\mathsf{HI}$).
3. Holds by definition of σ_0 and σ_1.
4. Follows from primeness and rules ($\mathsf{\vee I}_l$) and ($\mathsf{\vee I}_r$).
5. First, we show the inclusion from left to right. Assume $[\Gamma'] \in \sigma_0([X \vee Y])$, i.e., $\Gamma' \not\vdash X \vee Y$ and $\Gamma' \vdash \mathsf{H}(X \vee Y)$. We have $\Gamma' \not\vdash X$ and $\Gamma' \not\vdash Y$ by rules ($\mathsf{\vee I}_l$) and ($\mathsf{\vee I}_r$). Using rules ($\mathsf{\vee I}_l$), ($\mathsf{\vee I}_r$) and ($\mathsf{\vee HE}$) we obtain $\Gamma' \vdash X \vee Y \vee (\mathsf{H}X \wedge \mathsf{H}Y)$. Because $\Gamma' \not\vdash X$ and $\Gamma' \not\vdash Y$, by primeness of Γ' we have $\Gamma' \vdash \mathsf{H}X \wedge \mathsf{H}Y$. So $\Gamma' \vdash \mathsf{H}X$ and $\Gamma' \vdash \mathsf{H}Y$ by ($\mathsf{\wedge I}_l$) and ($\mathsf{\wedge I}_r$). Thus $[\Gamma'] \in \sigma_0([X]) \cap \sigma_0([Y])$.

   For the other inclusion, assume $[\Gamma'] \in \sigma_0([X]) \cap \sigma_0([Y])$. Then $\Gamma' \not\vdash X$, $\Gamma' \not\vdash Y$, $\Gamma' \vdash \mathsf{H}X$ and $\Gamma' \vdash \mathsf{H}Y$. Thus $\Gamma' \vdash \mathsf{H}(X \vee Y)$ by ($\mathsf{\vee HI}$). If $\Gamma' \vdash X \vee Y$ then $\Gamma' \vdash X$ or $\Gamma' \vdash Y$ by primeness of Γ', which gives a contradiction. Hence $\Gamma' \not\vdash X \vee Y$. Therefore $[\Gamma'] \in \sigma_0([X \vee Y])$.
6. Follows from rules ($\mathsf{\wedge I}$), ($\mathsf{\wedge E}_l$) and ($\mathsf{\wedge E}_r$).
7. For the implication from left to right, suppose $\Gamma' \vdash \mathsf{H}(X \wedge Y)$ but $\Gamma' \not\vdash X \wedge Y$. Using ($\mathsf{\wedge HE}$) we may show $\Gamma' \vdash \mathsf{H}X \vee \mathsf{H}Y$. By primeness of Γ' we obtain $\Gamma' \vdash \mathsf{H}X$ or $\Gamma' \vdash \mathsf{H}Y$. Since $\Gamma' \not\vdash X \wedge Y$, also $\Gamma' \not\vdash X$ or $\Gamma' \not\vdash Y$ by rule ($\mathsf{\wedge I}$). Without loss of generality assume $\Gamma' \not\vdash X$. Then $[\Gamma'] \in \sigma_0([X])$. Let $\Gamma'' \supseteq \Gamma'$ be prime and consistent with $\Gamma'' \vdash X$. We need to show $\Gamma'' \vdash \mathsf{H}Y$. But this follows from ($\mathsf{\wedge HE}_l$).

   For the other direction, suppose $\Gamma' \vdash \mathsf{H}X$ and $\Gamma' \not\vdash X$ and

   ($\star$) for all prime and consistent $\Gamma'' \supseteq \Gamma'$ such that $\Gamma'' \vdash X$ we have $\Gamma'' \vdash \mathsf{H}Y$.

   We would like to show $\Gamma', X \vdash \mathsf{H}Y$, and then conclude $\Gamma' \vdash \mathsf{H}(X \wedge Y)$ by rule ($\mathsf{\wedge HI}_l$), but $\Gamma' \cup \{X\}$ may not be prime. However, we can use Lemma 4.1.10. Suppose $\Gamma', X \not\vdash \mathsf{H}Y$. Then by Lemma 4.1.10 there exists a prime $\Gamma''' \supseteq \Gamma' \cup \{X\}$ such that $\Gamma'' \not\vdash \mathsf{H}Y$. But $\Gamma'' \vdash X$, which contradicts ($\star$). Hence ultimately $\Gamma', X \vdash \mathsf{H}Y$ and thus $\Gamma' \vdash \mathsf{H}(X \wedge Y)$ by rule ($\mathsf{\wedge HI}_l$). Also $\Gamma' \not\vdash X \wedge Y$, because otherwise $\Gamma' \vdash X$ by rule ($\mathsf{\wedge E}_l$).

8. The implication from left to right follows from rules (PE) and (PHE_l). The implication from right to left follows from rules (PI_l) and (PI_r) and from Lemma 4.1.10.
9. The implication from left to right follows from rules (PHE_l), (PHE_r), (PI_l) and (PI_r), and from Lemma 4.1.10. The implication from right to left follows from Lemma 4.1.10 and rules (PHI) and (PE).

We define the valuation ρ by $\rho(x) = [x]$. By Lemma 4.1.10 there exists a prime $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash M$. So $[\Gamma'] \in S$. It is easy to check that $[\Gamma'], \rho \Vdash \Gamma$ but $[\Gamma'], \rho \nVdash M$. □

Note that the above proof does not imply the consistency of $\mathcal{I}$Jp, because to construct the model we assume $\Gamma \nvdash M$.

4.1.2 Classical semantics

In this section we define classical semantics for $\mathcal{I}$Kp. It is in fact a restriction of the Kripke semantics for $\mathcal{I}$Jp to single-state Kripke models.

Definition 4.1.12. An $\mathcal{I}$Kp-*model* is a Kripke $\mathcal{I}$Jp-model with exactly one state s_0. For an $\mathcal{I}$Kp-model we adopt the abbreviations $\mathcal{T} = \{a \mid s_0 \in \sigma_1(a)\}$ and $\mathcal{F} = \{a \mid s_0 \in \sigma_0(a)\}$. Note that a PICA $\mathcal{C}$ and the sets $\mathcal{T}$ and $\mathcal{F}$ uniquely determine an $\mathcal{I}$Kp-model. We sometimes say that a tuple $\mathcal{M} = \langle \mathcal{C}, I, \mathcal{T}, \mathcal{F} \rangle$ is an $\mathcal{I}$Kp-model.

For convenience of reference, we reformulate in terms of $\mathcal{T}$ and $\mathcal{F}$ the conditions on σ_0 and σ_1 from Definition 4.1.5:

1. $\bot \in \mathcal{F}$,
2. $\mathcal{T} \cap \mathcal{F} = \emptyset$.
3. $\vee \cdot a \cdot b \in \mathcal{T}$ iff $a \in \mathcal{T}$ or $b \in \mathcal{T}$,
4. $\vee \cdot a \cdot b \in \mathcal{F}$ iff $a \in \mathcal{F}$ and $b \in \mathcal{F}$,
5. $\wedge \cdot a \cdot b \in \mathcal{T}$ iff $a \in \mathcal{T}$ and $b \in \mathcal{T}$,
6. $\wedge \cdot a \cdot b \in \mathcal{F}$ iff $a \in \mathcal{F}$ or $b \in \mathcal{F}$,
7. $\mathsf{p} \cdot a \cdot b \in \mathcal{T}$ iff $a \in \mathcal{F}$ or $b \in \mathcal{T}$,
8. $\mathsf{p} \cdot a \cdot b \in \mathcal{F}$ iff $a \in \mathcal{T}$ and $b \in \mathcal{F}$.

For an $\mathcal{I}$Kp-model $\mathcal{M}$, we use the notation $\mathcal{M}, \rho \models_i M$ or $\rho \models_i M$, instead of $s_0, \rho \Vdash_i M$. The notations $\mathcal{M}, \rho \models_i \Gamma$, $\rho \models_i \Gamma$, $\Gamma \models_i \mathcal{M}$, $\Gamma \models_{\mathcal{I}\mathrm{Kp}} M$ are defined in the obvious way.

Lemma 4.1.13. *In every $\mathcal{I}$Kp-model we have:*

- $\mathsf{h} \cdot a \in \mathcal{T}$ *iff* $a \in \mathcal{T} \cup \mathcal{F}$,
- $\neg \cdot a \in \mathcal{T}$ *iff* $a \in \mathcal{F}$,
- $\neg \cdot a \in \mathcal{F}$ *iff* $a \in \mathcal{T}$.

Proof. Follows directly from definitions. □

The intuitive interpretation of $\mathcal{T}$ and $\mathcal{F}$ is rather obvious: $\mathcal{T}$ is the set of true elements, and $\mathcal{F}$ is the set of false elements. An element is a proposition iff it is either true or false. Thus, by restricting our Kripke semantics for $\mathcal{I}$Jp to single-state models we obtain a quite natural semantics for $\mathcal{I}$Kp. We will now show that this semantics is sound and complete.

Theorem 4.1.14 (Soundness of classical semantics for $\mathcal{I}$Kp)**.**
If $\Gamma \vdash_{\mathcal{I}\mathrm{Kp}} M$ *then* $\Gamma \models_{\mathcal{I}\mathrm{Kp}} M$*. Moreover, if* $\Gamma \vdash_{\mathcal{I}\mathrm{Kp}} \neg M$ *then* $\Gamma \models_0 M$*.*

Proof. The proof of the first implication proceeds by induction on the length of derivation of $\Gamma \vdash M$, like in the proof of Theorem 4.1.8. Take any $\mathcal{I}$Kp-model $\mathcal{M}$ and any valuation ρ, and suppose $\mathcal{M}, \rho \models_1 \Gamma$. All axioms and rules except (EM) are checked by exactly the same proofs as in Theorem 4.1.8. So suppose the last rule in the derivation of $\Gamma \vdash M$ was (EM). Then $M \equiv X \vee \neg X$ and $\Gamma \vdash \mathsf{H}X$. Hence $\mathcal{M}, \rho \models_1 \mathsf{H}X$ by the IH. So $\mathcal{M}, \rho \models_1 X$ or $\mathcal{M}, \rho \models_0 X$, which implies $\mathcal{M}, \rho \models_1 M$.

Now suppose $\Gamma \vdash \neg M$. Then $\Gamma \models_1 \neg M$, which implies $\Gamma \models_0 M$. □

Definition 4.1.9 of primeness may be used for $\mathcal{I}$Kp if we interpret $\vdash$ there as provability in $\mathcal{I}$Kp. Then Lemma 4.1.10 also holds for $\mathcal{I}$Kp, by an identical proof.

Lemma 4.1.15. *If* Γ *is prime, then the following conditions hold:*

- $\Gamma \vdash X \vee Y$ *iff* $\Gamma \vdash X$ *or* $\Gamma \vdash Y$*,*
- $\Gamma \vdash \neg(X \vee Y)$ *iff* $\Gamma \vdash \neg X$ *and* $\Gamma \vdash \neg Y$*,*
- $\Gamma \vdash X \wedge Y$ *iff* $\Gamma \vdash X$ *and* $\Gamma \vdash Y$*,*
- $\Gamma \vdash \neg(X \wedge Y)$ *iff* $\Gamma \vdash \neg X$ *or* $\Gamma \vdash \neg Y$*,*
- $\Gamma \vdash X \supset Y$ *iff* $\Gamma \vdash \neg X$ *or* $\Gamma \vdash Y$*,*
- $\Gamma \vdash \neg(X \supset Y)$ *iff* $\Gamma \vdash X$ *and* $\Gamma \vdash \neg Y$*,*
- $\Gamma \vdash \neg\bot$*,*

where $\vdash$ *denotes provability in* $\mathcal{I}$Kp*.*

Proof. Easy, using Lemma 4.1.3. □

Theorem 4.1.16 (Completeness of classical semantics for $\mathcal{I}$Kp)**.**
If $\Gamma \models_{\mathcal{I}\mathrm{Kp}} M$ *then* $\Gamma \vdash_{\mathcal{I}\mathrm{Kp}} M$*. Moreover, if* $\Gamma \models_0 M$ *then* $\Gamma \vdash \neg M$*.*

Proof. The proof of the first implication is similar to the proof of Theorem 4.1.11, but easier. Assume $\Gamma \nvdash M$. By Lemma 4.1.10 there is a prime $\Gamma' \supseteq \Gamma$ with $\Gamma' \nvdash M$. We construct an $\mathcal{I}$Kp-model $\mathcal{M}$ like in the proof of Theorem 4.1.11, but as the single state we take Γ'. Note that with this construction we have:

- $[X] \in \mathcal{T}$ iff $\Gamma' \vdash X$,
- $[X] \in \mathcal{F}$ iff $\Gamma' \vdash \neg X$.

Using Lemma 4.1.15 it is easy to check the conditions from Definition 4.1.12. Then we take ρ such that $\rho(x) = [x]$, and check that $\mathcal{M}, \rho \models_1 \Gamma$ but $\mathcal{M}, \rho \not\models_1 M$.

Now suppose $\Gamma \models_0 M$. Then $\Gamma \models_1 \neg M$, and consequently $\Gamma \vdash \neg M$. □

$$\frac{}{\Gamma, X \vdash X}\ (\mathrm{Ax}) \qquad \frac{}{\Gamma \vdash \neg\bot}\ (\neg\bot)$$

$$\frac{\Gamma \vdash X \quad \Gamma \vdash Y}{\Gamma \vdash X \wedge Y}\ (\mathsf{\Lambda}\mathrm{I}) \qquad \frac{\Gamma \vdash X \wedge Y}{\Gamma \vdash X}\ (\mathsf{\Lambda}\mathrm{E}_l) \quad \frac{\Gamma \vdash X \wedge Y}{\Gamma \vdash Y}\ (\mathsf{\Lambda}\mathrm{E}_r)$$

$$\frac{\Gamma \vdash X}{\Gamma \vdash X \vee Y}\ (\mathsf{V}\mathrm{I}_l) \quad \frac{\Gamma \vdash Y}{\Gamma \vdash X \vee Y}\ (\mathsf{V}\mathrm{I}_r) \qquad \frac{\Gamma \vdash X \vee Y \quad \Gamma, X \vdash Z \quad \Gamma, Y \vdash Z}{\Gamma \vdash Z}\ (\mathsf{V}\mathrm{E})$$

$$\frac{\Gamma \vdash \neg X}{\Gamma \vdash \neg(X \wedge Y)}\ (\neg\mathsf{\Lambda}\mathrm{I}_l)$$

$$\frac{\Gamma \vdash \neg(X \wedge Y) \quad \Gamma, \neg X \vdash Z \quad \Gamma, \neg Y \vdash Z}{\Gamma \vdash Z}\ (\neg\mathsf{\Lambda}\mathrm{E})$$

$$\frac{\Gamma \vdash \neg Y}{\Gamma \vdash \neg(X \wedge Y)}\ (\neg\mathsf{\Lambda}\mathrm{I}_r)$$

$$\frac{\Gamma \vdash \neg(X \vee Y)}{\Gamma \vdash \neg X}\ (\neg\mathsf{V}\mathrm{E}_l)$$

$$\frac{\Gamma \vdash \neg X \quad \Gamma \vdash \neg Y}{\Gamma \vdash \neg(X \vee Y)}\ (\neg\mathsf{V}\mathrm{I})$$

$$\frac{\Gamma \vdash \neg(X \vee Y)}{\Gamma \vdash \neg Y}\ (\neg\mathsf{V}\mathrm{E}_r)$$

$$\frac{\Gamma \vdash X}{\Gamma \vdash \neg\neg X}\ (\neg\neg\mathrm{I}) \qquad \frac{\Gamma \vdash \neg\neg X}{\Gamma \vdash X}\ (\neg\neg\mathrm{E})$$

$$\frac{\Gamma \vdash X \quad X = Y}{\Gamma \vdash Y}\ (\mathrm{Eq}) \qquad \frac{\Gamma \vdash X \quad \Gamma \vdash \neg X}{\Gamma \vdash Y}\ (\neg\mathrm{E})$$

Figure 4.3: Rules of $\mathcal{I}\mathrm{Kp}'$

4.1.3 An alternative formulation of $\mathcal{I}\mathrm{Kp}$

Definition 4.1.17. The system $\mathcal{I}\mathrm{Kp}'$ has $\mathbb{T}(\Sigma)$ as the set of terms, with the signature Σ containing the illative constants: $\neg, \mathsf{\Lambda}, \mathsf{V}, \bot$. We adopt the abbreviations:

- $X \wedge Y \equiv \mathsf{\Lambda}XY$,
- $X \vee Y \equiv \mathsf{V}XY$,
- $X \supset Y \equiv \neg X \vee Y$,
- $\mathsf{H}X \equiv X \vee \neg X$.

The rules of $\mathcal{I}\mathrm{Kp}'$ are shown in Figure 4.3.

Formally, the language of $\mathcal{I}\mathrm{Kp}$ is different from the language of $\mathcal{I}\mathrm{Kp}'$, but there are obvious translations between the languages, so, e.g., the rules of $\mathcal{I}\mathrm{Kp}$ may be interpreted as rules with terms from the language of $\mathcal{I}\mathrm{Kp}'$, by replacing the constants of $\mathcal{I}\mathrm{Kp}$ with their translations in $\mathcal{I}\mathrm{Kp}'$. The following theorem implies that all rules from Figure 4.1 are derivable in $\mathcal{I}\mathrm{Kp}'$.

Theorem 4.1.18. *All rules of $\mathcal{I}\mathrm{Kp}'$ are derivable in $\mathcal{I}\mathrm{Kp}$. Conversely, all rules of $\mathcal{I}\mathrm{Kp}$ are derivable in $\mathcal{I}\mathrm{Kp}'$.*

Proof. First, we show that all rules of $\mathcal{I}\mathrm{Kp}'$ are derivable in $\mathcal{I}\mathrm{Kp}$. The rules (Ax), ($\wedge$I), ($\wedge$E$_l$), ($\wedge$E$_r$), ($\vee$I$_l$), ($\vee$I$_r$) and (Eq) are present in $\mathcal{I}\mathrm{Kp}$. The rule ($\neg\bot$) follows from ($\mathsf{H}\bot$I). The rules ($\neg\wedge$I$_l$), ($\neg\wedge$I$_r$), ($\neg\wedge$E), ($\neg\vee$I), ($\neg\vee$E$_l$), ($\neg\vee$E$_r$), ($\neg\neg$I) and ($\neg\neg$E) follow from Lemma 4.1.3. The rule ($\neg$E) follows from (PE) and ($\bot$E).

Now we show that all rules of $\mathcal{I}\mathrm{Kp}$ are derivable in $\mathcal{I}\mathrm{Kp}'$. The rules (Ax), ($\wedge$I), ($\wedge$E$_l$), ($\wedge$E$_r$), ($\vee$I$_l$), ($\vee$I$_r$) and (Eq) are present in $\mathcal{I}\mathrm{Kp}'$. The rule (EM) follows from the definition of H in $\mathcal{I}\mathrm{Kp}'$. We indicate how to derive the remaining rules.

($\bot\mathsf{HI}$) Follows from ($\neg\bot$) and ($\vee$I$_r$).

(PI_l) Follows from ($\vee$E), ($\vee$I$_l$) and ($\vee$I$_r$).

(PI_r) Follows from ($\vee$I$_r$).

(PE) Follows from ($\vee$E) and ($\neg$E).

(PHI) Follows from ($\vee$E), ($\vee$I$_l$), ($\vee$I$_r$), ($\neg\neg$I) and ($\neg\vee$I).

(PHE$_l$) Follows from ($\vee$E), ($\vee$I$_l$), ($\vee$I$_r$), ($\neg\vee$E$_l$) and ($\neg\neg$E).

(PHE$_r$) Follows from ($\vee$E), ($\neg\vee$E$_l$), ($\neg\neg$E) and ($\neg$E).

($\wedge\mathsf{HI}_l$) Follows from ($\vee$E), ($\vee$I$_l$), ($\vee$I$_r$), ($\wedge$I), ($\neg\wedge$I$_l$) and ($\neg\wedge$I$_r$).

($\wedge\mathsf{HI}_r$) Analogous to ($\wedge\mathsf{HI}_l$).

($\wedge\mathsf{H}$E) Follows from ($\vee$E), ($\vee$I$_l$), ($\vee$I$_r$), ($\wedge$E$_l$) and ($\neg\wedge$E).

($\wedge\mathsf{H}$E$_l$) Follows from ($\vee$E), ($\vee$I$_l$), ($\vee$I$_r$), ($\wedge$E$_r$), ($\neg\wedge$E) and ($\neg$E).

($\wedge\mathsf{H}$E$_r$) Analogous to ($\wedge\mathsf{H}$E$_l$).

($\vee\mathsf{HI}$) Follows from ($\vee$E), ($\vee$I$_l$), ($\vee$I$_r$) and ($\neg\vee$I).

($\vee\mathsf{H}$E) Follows from ($\vee$E), ($\vee$I$_l$), ($\vee$I$_r$), ($\neg\vee$E$_l$) and ($\neg\vee$E$_r$).

(HI) Follows from ($\vee$I$_l$).

($\bot$E) Follows from ($\neg\bot$) and ($\neg$E).

□

Lemma 4.1.19. *The following rule is admissible in $\mathcal{I}\mathrm{Kp}'$.*

$$\frac{\Gamma, \neg X \vdash \bot \quad \Gamma \vdash \mathsf{H}X}{\Gamma \vdash X}\ (\neg\mathrm{I})$$

Proof. Use (PI_l), ($\vee$E), ($\neg\neg$E) and ($\bot$E). □

4.2 Model constructions

In this section we construct models for $\mathcal{I}$Jp and $\mathcal{I}$Kp. A corollary of the model constructions is consistency of $\mathcal{I}$Jp and $\mathcal{I}$Kp. The constructions will also be used in the next section to prove completeness of translations of NJp into $\mathcal{I}$Jp, and of NKp into $\mathcal{I}$Kp.

To facilitate completeness of translation proofs, each construction of a model $\mathcal{M}$ for an illative system (including the ones in the subsequent chapters) will be parameterised by a model $\mathcal{N}$ for a corresponding traditional system, and there will be a natural injection from the set of elements true in (a state of) $\mathcal{N}$ to the set of elements true in (a corresponding state of) $\mathcal{M}$. We will give only two constructions for intuitionistic systems – the following one for $\mathcal{I}$Jp and one for a first-order system in Chapter 5. It is more difficult for intuitionistic than classical systems to construct a model with an appropriate injection which can be used in a completeness of translation proof.[2]

4.2.1 Model construction for $\mathcal{I}$Jp

Fix a Kripke NJp-model $\mathcal{S} = \langle S, \leq, \Vdash \rangle$. Our construction of a Kripke $\mathcal{I}$Jp-model $\mathcal{M}_\mathcal{S}$ will be parameterised by $\mathcal{S}$. As the states of $\mathcal{M}_\mathcal{S}$ we will adopt the states of $\mathcal{S}$. The Kripke $\mathcal{I}$Jp-model $\mathcal{M}_\mathcal{S}$ will be constructed in such a way that for each state s there will be a natural injection from the set of elements true in state s in $\mathcal{S}$ to the set of elements true in state s in $\mathcal{M}_\mathcal{S}$. In Section 4.3 this will be used to show completeness of a translation of NJp into $\mathcal{I}$Jp.

We assume that all elements of V_P, i.e., the propositional variables of NJp, are present as constants in the syntax of $\mathcal{I}$Jp. We adopt the abbreviation $\top \equiv \mathsf{P}\bot\bot$.

Definition 4.2.1. For $s \in S$ and an ordinal α we define binary relations $\succ_s^\alpha$ on $\mathbb{T}$ by induction. In the following the notation $X \rightsquigarrow_s^\alpha Y$ stands for $X \xrightarrow{*}_{\beta\eta} \cdot \succ_s^\alpha Y$, and $X \rightsquigarrow_s^{<\alpha} Y$ abbreviates "there is $\beta < \alpha$ with $X \rightsquigarrow_s^\beta Y$", and similarly $X \succ_s^{<\alpha} Y$ abbreviates "there is $\beta < \alpha$ with $X \succ_s^\beta Y$".

($V_\top$) $p \succ_s^\alpha \top$ if $p \in V_P$ and $\mathcal{S}, s \Vdash p$,

($V_\bot$) $p \succ_s^\alpha \bot$ if $p \in V_P$ and $\mathcal{S}, s \nVdash p$,

($\top_\top$) $\top \succ_s^\alpha \top$,

($\bot_\bot$) $\bot \succ_s^\alpha \bot$,

($\mathsf{V}_\top$) $X \vee Y \succ_s^\alpha \top$ if $X \rightsquigarrow_s^{<\alpha} \top$ or $Y \rightsquigarrow_s^{<\alpha} \top$,

($\mathsf{V}_\bot$) $X \vee Y \succ_s^\alpha \bot$ if $X \rightsquigarrow_s^{<\alpha} \bot$ and $Y \rightsquigarrow_s^{<\alpha} \bot$,

($\mathsf{\Lambda}_\top$) $X \wedge Y \succ_s^\alpha \top$ if $X \rightsquigarrow_s^{<\alpha} \top$ and $Y \rightsquigarrow_s^{<\alpha} \top$,

[2] Of course, if we do not require such an injection, then it is easier to construct models for intuitionistic systems, because every model for a classical system is a model for its intuitionistic version. But a model construction for a classical system cannot then be used to show completeness of a translation of traditional intuitionistic logic into intuitionistic illative combinatory logic.

$(\wedge_\bot)$ $X \wedge Y \succ_s^\alpha \bot$ if

- $X \rightsquigarrow_s^{<\alpha} \bot$ and for all $s' \geq s$, such that $X \rightsquigarrow_{s'}^{<\alpha} \top$, we have $Y \rightsquigarrow_{s'}^{<\alpha} \rho$ with $\rho \in \{\top, \bot\}$, or
- $Y \rightsquigarrow_s^{<\alpha} \bot$ and for all $s' \geq s$, such that $Y \rightsquigarrow_{s'}^{<\alpha} \top$, we have $X \rightsquigarrow_{s'}^{<\alpha} \rho$ with $\rho \in \{\top, \bot\}$,

$(\mathsf{P}_\top)$ $X \supset Y \succ_s^\alpha \top$ if

- $X \rightsquigarrow_s^{<\alpha} \rho$ with $\rho \in \{\top, \bot\}$, and for every $s' \geq s$ such that $X \rightsquigarrow_{s'}^{<\alpha} \top$ we have $Y \rightsquigarrow_{s'}^{<\alpha} \top$, or
- $Y \rightsquigarrow_s^{<\alpha} \top$.

$(\mathsf{P}_\bot)$ $X \supset Y \succ_s^\alpha \bot$ if

- $X \rightsquigarrow_s^{<\alpha} \rho$ with $\rho \in \{\top, \bot\}$, and
- for every $s' \geq s$ such that $X \rightsquigarrow_{s'}^{<\alpha} \top$ we have $Y \rightsquigarrow_{s'}^{<\alpha} \rho$ with $\rho \in \{\top, \bot\}$, and
- there exists $s' \geq s$ such that $X \rightsquigarrow_{s'}^{<\alpha} \top$ and $Y \rightsquigarrow_{s'}^{<\alpha} \bot$.

Above X and Y are arbitrary terms.

Note that it is not obvious that $\succ_s^\alpha \subseteq \succ_s^\beta$ for $\alpha \leq \beta$, because of the negative conditions in $(\wedge_\bot)$, $(\mathsf{P}_\top)$ and $(\mathsf{P}_\bot)$. We will show this only in Lemma 4.2.5. However, we obviously have $\succ_s^{<\alpha} \subseteq \succ_s^{<\beta}$ and $\rightsquigarrow_s^{<\alpha} \subseteq \rightsquigarrow_s^{<\beta}$ for $\alpha \leq \beta$.

In the rest of this section we assume that $s \in \mathcal{S}$, $p \in V_P$, $\rho, \rho', \ldots \in \{\top, \bot\}$ and M, N, X, Y, Z, etc., are terms of $\mathcal{I}$Jp, unless otherwise stated.

Lemma 4.2.2. *If $X \succ_s^\alpha \rho$ and $X \stackrel{*}{\to}_{\beta\eta} Y$, then $Y \succ_s^\alpha \rho$.*

Proof. Induction on α.

First, notice that the inductive hypothesis implies:

($\star$) for all terms M, N and all $\rho \in \{\top, \bot\}$, if $M \rightsquigarrow_s^{<\alpha} \rho$ and $M \stackrel{*}{\to}_{\beta\eta} N$, then $N \rightsquigarrow_s^{<\alpha} \rho$.

Indeed, assume $M \rightsquigarrow_s^{<\alpha} \rho$ and $M \stackrel{*}{\to}_{\beta\eta} N$. Then $M \stackrel{*}{\to}_{\beta\eta} M' \succ_s^{<\alpha} \rho$ for some M'. By confluence of $\lambda\beta\eta$ there is N' such that $N \stackrel{*}{\to}_{\beta\eta} N'$ and $M' \stackrel{*}{\to}_{\beta\eta} N'$. By the inductive hypothesis $N' \succ_s^{<\alpha} \rho$. Thus $N \rightsquigarrow_s^{<\alpha} \rho$. See Figure 4.4.

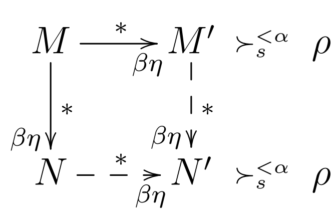

Figure 4.4

Now assume $X \succ_s^\alpha \rho$ and $X \stackrel{*}{\to}_{\beta\eta} Y$. We need to consider all possible rules by which $X \succ_s^\alpha \rho$ may be obtained.

Suppose $X \succ_s^\alpha \rho$ follows by rule $(\mathsf{P}_\top)$. Then $\rho \equiv \top$, $X \equiv X_1 \supset X_2$, $Y \equiv Y_1 \supset Y_2$, $X_i \xrightarrow{*}_{\beta\eta} Y_i$, and $X_2 \leadsto_s^{<\alpha} \top$ or:

(a) $X_1 \leadsto_s^{<\alpha} \rho'$ with $\rho' \in \{\top, \bot\}$, and

(b) for every $s' \geq s$ such that $X_1 \leadsto_{s'}^{<\alpha} \top$ we have $X_2 \leadsto_{s'}^{<\alpha} \top$.

If $X_2 \leadsto_s^{<\alpha} \top$ then also $Y_2 \leadsto_s^{<\alpha} \top$ by $(\star)$. Thus $Y \succ_s^\alpha \top$. So assume (a) and (b) hold. By $(*)$ the condition (a) still holds with Y_1 substituted for X_1. Assume $s' \geq s$ and $Y_1 \leadsto_{s'}^{<\alpha} \top$. Then also $X_1 \leadsto_{s'}^{<\alpha} \top$ because $X_1 \xrightarrow{*}_{\beta\eta} Y_1$. So $X_2 \leadsto_{s'}^{<\alpha} \top$ by (b). Thus by $(\star)$ we have $Y_2 \leadsto_{s'}^{<\alpha} \top$. Hence, (b) holds with Y_i substituted for X_i. Therefore, $Y \succ_s^\alpha \top$.

Suppose $X \succ_s^\alpha \rho$ follows by rule $(\mathsf{P}_\bot)$. Then $\rho \equiv \bot$, $X \equiv X_1 \supset X_2$ and $Y \equiv Y_1 \supset Y_2$ with $X_i \xrightarrow{*}_{\beta\eta} Y_i$. Also:

(a) $X_1 \leadsto_s^{<\alpha} \rho'$ with $\rho' \in \{\top, \bot\}$, and

(b) for every $s' \geq s$ such that $X_1 \leadsto_{s'}^{<\alpha} \top$ we have $X_2 \leadsto_{s'}^{<\alpha} \rho'$ with $\rho' \in \{\top, \bot\}$, and

(c) there exists $s' \geq s$ such that $X_1 \leadsto_{s'}^{<\alpha} \top$ and $X_2 \leadsto_{s'}^{<\alpha} \bot$.

By $(\star)$ the condition (a) still holds for Y_1. Suppose $s' \geq s$ and $Y_1 \leadsto_{s'}^{<\alpha} \top$. Then also $X_1 \leadsto_{s'}^{<\alpha} \top$, because $X_1 \xrightarrow{*}_{\beta\eta} Y_1$. Hence $X_2 \leadsto_{s'}^{<\alpha} \rho'$ with $\rho' \in \{\top, \bot\}$. By $(\star)$ also $Y_2 \leadsto_{s'}^{<\alpha} \rho'$. So (b) holds for Y, i.e., with Y_i substituted for X_i. Let $s' \geq s$ be such that $X_1 \leadsto_{s'}^{<\alpha} \top$ and $X_2 \leadsto_{s'}^{<\alpha} \bot$. By $(\star)$ we have $Y_1 \leadsto_{s'}^{<\alpha} \top$ and $Y_2 \leadsto_{s'}^{<\alpha} \bot$. Thus (c) holds for Y. Therefore, $Y \succ_s^\alpha \bot$.

Other cases are similar. □

Corollary 4.2.3. *$X \leadsto_s^\alpha Y$ iff there exists X' such that $X =_{\beta\eta} X' \succ_s^\alpha Y$.*

Lemma 4.2.4. *The following conditions hold.*

1. *If $M \succ_s^\alpha \top$ and $s \leq s_0$ then $M \succ_{s_0}^\alpha \top$.*
2. *If $M \succ_s^\alpha \bot$ and $s \leq s_0$ then $M \succ_{s_0}^\alpha \top$ or $M \succ_{s_0}^\alpha \bot$.*

Proof. Induction on α.

1. Follows directly from definitions and the inductive hypothesis.
2. Suppose $M \succ_s^\alpha \bot$ and $s_0 \geq s$. The only non-obvious cases are when $(\mathsf{\Lambda}_\bot)$ or $(\mathsf{P}_\bot)$ is used to obtain $M \succ_s^\alpha \bot$.

   $(\mathsf{\Lambda}_\bot)$ Then $M \equiv X \wedge Y$ and e.g.

   $(\star)$ $X \leadsto_s^{<\alpha} \bot$ and for all $s' \geq s$, such that $X \leadsto_{s'}^{<\alpha} \top$, we have $Y \leadsto_{s'}^{<\alpha} \rho$ with $\rho \in \{\top, \bot\}$.

   If $X \leadsto_{s_0}^{<\alpha} \bot$, then still $M \succ_{s_0}^\alpha \bot$. If not, then by the second part of the inductive hypothesis $X \leadsto_{s_0}^{<\alpha} \top$. By $(\star)$ we have $Y \leadsto_{s_0}^{<\alpha} \top$ or $Y \leadsto_{s_0}^{<\alpha} \bot$. If $Y \leadsto_{s_0}^{<\alpha} \top$ then $M \succ_{s_0}^\alpha \top$ by $(\mathsf{\Lambda}_\top)$. Otherwise, by the first part of the inductive hypothesis, $X \leadsto_{s'}^{<\alpha} \top$ for all $s' \geq s_0$. So in particular for all $s' \geq s_0$ such that $Y \leadsto_{s'}^{<\alpha} \top$ we have $X \leadsto_{s'}^{<\alpha} \top$. Thus $M \succ_{s_0}^\alpha \bot$ by $(\mathsf{\Lambda}_\bot)$.

   $(\mathsf{P}_\bot)$ Then $M \equiv X \supset Y$ and

(a) $X \rightsquigarrow_s^{<\alpha} \rho$ with $\rho \in \{\top, \bot\}$, and
(b) for every $s' \geq s$ such that $X \rightsquigarrow_{s'}^{<\alpha} \top$ we have $Y \rightsquigarrow_{s'}^{<\alpha} \rho$ with $\rho \in \{\top, \bot\}$, and
(c) there exists $s' \geq s$ such that $X \rightsquigarrow_{s'}^{<\alpha} \top$ and $Y \rightsquigarrow_{s'}^{<\alpha} \bot$.

By the inductive hypothesis, $X \rightsquigarrow_{s_0}^{<\alpha} \rho'$ for some $\rho' \in \{\top, \bot\}$, so (a) holds with s_0 instead of s. The condition (b) also holds with s_0 substituted for s. Assume (c) does not hold for s_0, i.e., for every $s' \geq s_0$ such that $X \rightsquigarrow_{s'}^{<\alpha} \top$ we have $Y \not\rightsquigarrow_{s'}^{<\alpha} \bot$. Then by (b), for every $s' \geq s_0$ such that $X \rightsquigarrow_{s'}^{<\alpha} \top$ we have $Y \rightsquigarrow_{s'}^{<\alpha} \top$. Therefore, $M \succ_{s_0}^{\alpha} \top$ by $(\mathsf{P}_\top)$.

□

Lemma 4.2.5. *The following conditions hold.*

1. *If* $M \succ_s^{<\alpha} \rho$ *then* $M \succ_s^{\alpha} \rho$.
2. *If* $M \succ_s^{\alpha} \top$ *then* $M \not\succ_s^{\alpha} \bot$.

Proof. Induction on α.

First, note that it follows from the inductive hypothesis that:

($\star$) for $\gamma < \alpha$, if $X \rightsquigarrow_s^{\gamma} \top$ then $X \not\rightsquigarrow_s^{\gamma} \bot$.

Indeed, assume $X \rightsquigarrow_s^{\gamma} \top$ and $X \rightsquigarrow_s^{\gamma} \bot$. Then $X \xrightarrow{*}_{\beta\eta} X_1 \succ_s^{\gamma} \top$ and $X \xrightarrow{*}_{\beta\eta} X_2 \succ_s^{\gamma} \bot$. By confluence of $\lambda\beta\eta$, there is Y such that $X_1 \xrightarrow{*}_{\beta\eta} Y$ and $X_2 \xrightarrow{*}_{\beta\eta} Y$. By Lemma 4.2.2 we obtain $Y \succ_s^{\gamma} \top$ and $Y \succ_s^{\gamma} \bot$. This contradicts the second part of the inductive hypothesis. See Figure 4.5.

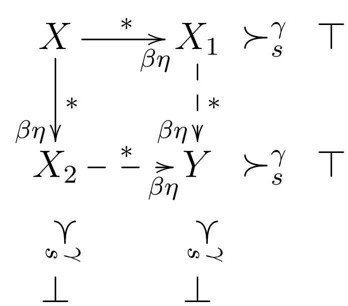

Figure 4.5

Now we check conditions 1 and 2.

1. Suppose, e.g., that $X \supset Y \succ_s^{\beta} \top$ is obtained by $(\mathsf{P}_\top)$ for some $\beta < \alpha$. The other cases are similar. We want to show $X \supset Y \succ_s^{\alpha} \top$. We have $Y \rightsquigarrow_s^{<\beta} \top$ or

   (a) $X \rightsquigarrow_s^{<\beta} \rho$ with $\rho \in \{\top, \bot\}$, and
   (b) for every $s' \geq s$ such that $X \rightsquigarrow_{s'}^{<\beta} \top$ we have $Y \rightsquigarrow_{s'}^{<\beta} \top$.

   If $Y \rightsquigarrow_s^{<\beta} \top$ then also $Y \rightsquigarrow_s^{<\beta} \top$, so $X \supset Y \succ_s^{\alpha} \top$.

Thus assume (a) and (b) hold. Obviously, (a) still holds with α instead of β. So assume $s' \geq s$ and $X \leadsto_{s'}^{<\alpha} \top$, i.e., $X \leadsto_{s'}^{\gamma} \top$ for some $\gamma < \alpha$.

If $\gamma < \beta$ then obviously $X \leadsto_{s'}^{<\beta} \top$. Assume $\beta \leq \gamma < \alpha$. Since $X \leadsto_{s}^{<\beta} \rho$, by Lemma 4.2.4 we have $X \leadsto_{s'}^{<\beta} \top$ or $X \leadsto_{s'}^{<\beta} \bot$. If $X \leadsto_{s'}^{<\beta} \bot$ then $X \leadsto_{s'}^{\gamma} \bot$ by the first part of the inductive hypothesis. However, this contradicts ($\star$), because also $X \leadsto_{s'}^{\gamma} \top$.

Therefore $X \leadsto_{s'}^{<\beta} \top$. Then $Y \leadsto_{s'}^{<\beta} \top$ by (b), so also $Y \leadsto_{s'}^{<\alpha} \top$. Thus (b) holds with α instead of β. This proves that $X \supset Y \succ_{s}^{\alpha} \top$.

2. Follows from ($\star$) and Definition 4.2.1.

□

Lemma 4.2.5 implies that $\succ_{s}^{\alpha} \subseteq \succ_{s}^{\beta}$ for $\alpha \leq \beta$ and $s \in \mathcal{S}$. Therefore, by Theorem 2.1.3 there exists the closure ordinal of Definition 4.2.1, i.e., the least ordinal ζ such that $\succ_{s}^{\zeta} = \succ_{s}^{<\zeta}$ for each $s \in \mathcal{S}$. We write $\succ_s$ and $\leadsto_s$ without superscripts to denote $\succ_{s}^{\zeta}$ and $\leadsto_{s}^{\zeta}$. It is not difficult to check that if the set of states $\mathcal{S}$ is finite then $\zeta = \omega$. In general, the closure ordinal may depend on the cardinality of $\mathcal{S}$.

Note that Lemma 4.2.2 and the second part of Lemma 4.2.5 imply the following corollary.

Corollary 4.2.6. *The reduction system $\langle \to_{\beta\eta}, \{\succ_s\}_{s \in \mathcal{S}} \rangle$ is coherent.*

Now, we are ready to construct the model $\mathcal{M}$ for $\mathcal{I}$Jp.

Definition 4.2.7. Define $\mathcal{M}_{\mathcal{S}} = \langle \mathcal{C}, I, S, \leq, \sigma_0, \sigma_1 \rangle$ where:

- $\mathcal{C}$ is the extensional propositional illative combinatory algebra constructed from the $\beta\eta$-equality equivalence classes of terms, with $\mathsf{k} = [\mathsf{K}]$, $\mathsf{s} = [\mathsf{S}]$, $\mathsf{p} = [\mathsf{P}]$, etc., where by $[X]$ we denote the equivalence class of X,
- I is defined by $I(c) = [c]$ for $c \in \Sigma$,
- S is the set of states of $\mathcal{S}$,
- $\leq$ is the order on states from $\mathcal{S}$,
- $\sigma_1([X]) = \{s \in S \mid X \leadsto_s \top\}$,
- $\sigma_0([X]) = \{s \in S \mid X \leadsto_s \bot\}$.

Note that by coherence σ_0 and σ_1 are well-defined, i.e., the definitions do not depend on the choice of representants.

Theorem 4.2.8. *The structure $\mathcal{M}_{\mathcal{S}}$ is a Kripke $\mathcal{I}$Jp-model such that for each $p \in V_P$ there is $\overline{p} \in \mathcal{C}$ satisfying for each $s \in S$:*

- *$s \in \sigma_1(\overline{p})$ iff $\mathcal{S}, s \Vdash p$,*
- *$s \in \sigma_0(\overline{p})$ iff $\mathcal{S}, s \nVdash p$.*

Proof. Using Corollary 4.2.6 it is easy to verify the conditions from Definition 4.1.5. The additional requirements in the statement of the theorem follow directly from definitions. □

Corollary 4.2.9. *The system* $\mathcal{I}$Jp *is consistent, i.e.,* $\nvdash_{\mathcal{I}\mathrm{Jp}} \bot$.

Proof. Since there exists some Kripke NJp-model, by Theorem 4.2.8 there exists a Kripke $\mathcal{I}$Jp-model $\mathcal{M}$. If $\vdash_{\mathcal{I}\mathrm{Jp}} \bot$, then $\mathcal{M} \Vdash \bot$ by Theorem 4.1.8. This is a contradiction. □

4.2.2 Model construction for $\mathcal{I}$Kp

In this section we give a model construction for $\mathcal{I}$Kp. Essentially, it is a simplification of the construction for $\mathcal{I}$Jp.

Let v be an NKp-valuation. Our construction of an $\mathcal{I}$Kp-model $\mathcal{M}_v$ will be parameterised by v. We assume that the propositional variables from V_P are present as constants in the set of terms $\mathbb{T}$.

Definition 4.2.10. We define a binary relation $\succ \subseteq \mathbb{T} \times \mathbb{T}$ inductively:

($V_\top$) $p \succ \top$ if $p \in V_p$ and $v(p) = 1$,

($V_\bot$) $p \succ \bot$ if $p \in V_p$ and $v(p) = 0$,

($\top_\top$) $\top \succ \top$,

($\bot_\bot$) $\bot \succ \bot$,

($\mathsf{V}_\top$) $X \vee Y \succ \top$ if $X \succ \top$ or $Y \succ \top$,

($\mathsf{V}_\bot$) $X \vee Y \succ \bot$ if $X \succ \bot$ and $Y \succ \bot$,

($\mathsf{\Lambda}_\top$) $X \wedge Y \succ \top$ if $X \succ \top$ and $Y \succ \top$,

($\mathsf{\Lambda}_\bot$) $X \wedge Y \succ \bot$ if $X \succ \bot$ or $Y \succ \bot$,

($\mathsf{P}_\top$) $X \supset Y \succ \top$ if $X \succ \bot$ or $Y \succ \top$,

($\mathsf{P}_\bot$) $X \supset Y \succ \bot$ if $X \succ \top$ and $Y \succ \bot$.

Like in Section 2.1, we use $\succ^\alpha$ to denote the α-th approximant of $\succ$, and we set $\succ^{<\alpha} = \bigcup_{\beta<\alpha} \succ^\alpha$. As in Section 4.2.1 we use $X, Y, M, \ldots$ for terms. The closure ordinal of the definition of $\succ$ is clearly ω.

Lemma 4.2.11. *The reduction system* $\langle \to_{\beta\eta}, \succ \rangle$ *is coherent.*

Proof. We check the conditions in Definition 2.3.1.

1. $\to_{\beta\eta}$ is confluent by Theorem 2.3.9.
2. It follows by straightforward induction on α that $\to_{\beta\eta}$ preserves $\succ^\alpha$. For instance, assume $X \vee Y \succ^\alpha \top$ and $X \vee Y \xrightarrow{*}_{\beta\eta} X' \vee Y'$. Then e.g. $X \succ^{<\alpha} \top$. By the IH we have $X' \succ^{<\alpha} \top$. Thus $X' \vee Y' \succ^\alpha \top$.
3. By straightforward induction on α one shows that if $X \succ^\alpha \top$ then $X \not\succ^\alpha \bot$.

□

Definition 4.2.12. Define $\mathcal{M}_v = \langle \mathcal{C}, I, \mathcal{T}, \mathcal{F} \rangle$ where:

- $\mathcal{C}$ is the extensional propositional illative combinatory algebra constructed from the $\beta\eta$-equality equivalence classes of terms, with $\mathsf{k} = [\mathsf{K}]$, $\mathsf{s} = [\mathsf{S}]$, $\mathsf{p} = [\mathsf{P}]$, etc., where by $[X]$ we denote the equivalence class of X,
- I is defined by $I(c) = [c]$ for $c \in \Sigma$,
- $\mathcal{T} = \{[X] \mid X \rightsquigarrow \top\}$,
- $\mathcal{F} = \{[X] \mid X \rightsquigarrow \bot\}$.

Here $\rightsquigarrow$ is defined as usual: $X \rightsquigarrow Y$ iff $X \xrightarrow{*}_{\beta\eta} \cdot \succ Y$.

Theorem 4.2.13. *The structure $\mathcal{M}_v$ is an $\mathcal{I}\mathrm{Kp}$-model such that for every $p \in V_P$ there is $\overline{p} \in \mathcal{C}$ satisfying:*

- *$\overline{p} \in \mathcal{T}$ iff $v(p) = 1$,*
- *$\overline{p} \in \mathcal{F}$ iff $v(p) = 0$.*

Proof. Using Lemma 4.2.11, it is easy to check the conditions from Definition 4.1.12. The additional condition in the statement of the theorem holds by construction. □

Corollary 4.2.14. *The system $\mathcal{I}\mathrm{Kp}$ is consistent, i.e., $\nvdash_{\mathcal{I}\mathrm{Kp}} \bot$.*

4.3 Translations

In this section we prove that there exist sound and complete syntactic translations of traditional systems of propositional logic into the corresponding illative systems. The proofs are done semantically, using the results of the previous section. Since the translations are very straightforward and natural, the results of this section may be seen as establishing conservativity of propositional illative systems over the corresponding traditional systems.

Translations very similar to the ones we provide, both for propositional logic and for first-order logic, were already defined before. Their soundness was shown syntactically. See [Bun74a, BBD93].

We adopt the notational conventions like in the previous section, i.e., X, Y, Z stand for terms in $\mathbb{T}$, etc. Also φ, ψ, etc., stand for propositional formulas, and Δ, Δ', etc., stand for sets of propositional formulas. We assume that all propositional variables from V_P occur as constants in $\mathbb{T}$. Sometimes we write, e.g., Δ, φ instead of $\Delta \cup \{\varphi\}$.

Definition 4.3.1. We define a mapping $\lceil - \rceil$ from propositional formulas to the set of terms $\mathbb{T}$ of illative systems, and a context-providing mapping $\Gamma(-)$ from sets of propositional formulas to sets of terms. The definition of $\lceil \varphi \rceil$ is by induction on the structure of φ:

- $\lceil p \rceil \equiv p$ for $p \in V_P$,
- $\lceil \bot \rceil \equiv \bot$,
- $\lceil \varphi \vee \psi \rceil \equiv \lceil \varphi \rceil \vee \lceil \psi \rceil$,
- $\lceil \varphi \wedge \psi \rceil \equiv \lceil \varphi \rceil \wedge \lceil \psi \rceil$,

- $\lceil \varphi \to \psi \rceil \equiv \lceil \varphi \rceil \supset \lceil \psi \rceil$.

We extend the mapping $\lceil - \rceil$ to sets of propositional formulas thus: $\lceil \Delta \rceil = \{\lceil \varphi \rceil \mid \varphi \in \Delta\}$.

For a set of propositional formulas Δ, the set $\Gamma(\Delta)$ is defined to contain $\mathsf{H}p$ for each $p \in \mathrm{FV}(\Delta)$.

Note that, e.g., in the right-hand side of the third rule for $\lceil - \rceil$ the expression $\lceil \varphi \rceil \vee \lceil \psi \rceil$ is just an abbreviation for $\mathsf{V}\lceil \varphi \rceil \lceil \psi \rceil$, whereas the $\vee$ in the left-hand side is an operator in the syntax of propositonal formulas.

The mapping $\Gamma(-)$ provides so-called "grammatical conditions". In illative systems it is not specified a priori which category a given variable belongs to, i.e., what is the type of the variable. So this information must be provided explicitly in the context.

Lemma 4.3.2. $\Gamma(\{\varphi\}) \vdash_{\mathcal{I}\mathrm{Jp}} \mathsf{H}\lceil \varphi \rceil$.

Proof. Induction on the structure of φ. □

Theorem 4.3.3 (Completeness of the translation for $\mathcal{I}$Jp)**.**
$\Delta \Vdash_{\mathrm{NJp}} \varphi$ *iff* $\Gamma(\Delta, \varphi), \lceil \Delta \rceil \Vdash_{\mathcal{I}\mathrm{Jp}} \lceil \varphi \rceil$.

Proof. Assume $\Delta \Vdash_{\mathrm{NJp}} \varphi$. Let $\mathcal{M}$ be a Kripke $\mathcal{I}$Jp-model, s_0 a state of $\mathcal{M}$, and ρ an $\mathcal{M}$-valuation such that $\mathcal{M}, s_0, \rho \Vdash \Gamma(\Delta, \varphi), \lceil \Delta \rceil$. We define a Kripke NJp-model $\mathcal{S} = \langle S, \leq, \Vdash \rangle$ by taking S and $\leq$ to be the same as in $\mathcal{M}$, and defining $\Vdash$ by: $\mathcal{S}, s \Vdash p$ iff $\mathcal{M}, s, \rho \Vdash p$. By induction on the structure of a subformula ψ of a formula from $\Delta \cup \{\varphi\}$, it is easy to prove that for $s \geq s_0$ we have: $\mathcal{S}, s \Vdash \psi$ iff $\mathcal{M}, s, \rho \Vdash \lceil \psi \rceil$. By way of an example, we show the case $\psi \equiv \psi_1 \to \psi_2$. Other cases are similar. We have $\lceil \psi \rceil \equiv \lceil \psi_1 \rceil \supset \lceil \psi_2 \rceil$.

Suppose $\mathcal{S}, s \Vdash \psi_1 \to \psi_2$. Then for every $s' \geq s$ such that $\mathcal{S}, s' \Vdash \psi_1$ we have $\mathcal{S}, s' \Vdash \psi_2$. Let $s' \geq s$ be such that $\mathcal{M}, s', \rho \Vdash \lceil \psi_1 \rceil$. By the induction hypothesis $\mathcal{S}, s' \Vdash \psi_1$, so $\mathcal{S}, s' \Vdash \psi_2$. Applying again the induction hypothesis we obtain $\mathcal{M}, s', \rho \Vdash \lceil \psi_2 \rceil$. By Lemma 4.3.2 we also have $\Gamma(\psi_1) \vdash_{\mathcal{I}\mathrm{Jp}} \mathsf{H}\lceil \psi_1 \rceil$. Since ψ_1 is a subformula of a formula from $\Delta \cup \{\varphi\}$ and $\mathcal{M}, s, \rho \Vdash \Gamma(\Delta, \varphi)$, we obtain $\mathcal{M}, s, \rho \Vdash \Gamma(\psi_1)$. Hence $\mathcal{M}, s, \rho \Vdash \mathsf{H}\lceil \psi_1 \rceil$ by Theorem 4.1.8. Therefore, we finally conclude $\mathcal{M}, s, \rho \Vdash \lceil \psi \rceil$ by 5 in Lemma 4.1.7.

Now assume $\mathcal{M}, s, \rho \Vdash \lceil \psi_1 \rceil \supset \lceil \psi_2 \rceil$. Let $s' \geq s$ be such that $\mathcal{S}, s' \Vdash \psi_1$. By the inductive hypothesis $\mathcal{M}, s', \rho \Vdash \lceil \psi_1 \rceil$, so $\mathcal{M}, s', \rho \Vdash \lceil \psi_2 \rceil$. Again by the inductive hypothesis $\mathcal{S}, s' \Vdash \psi_2$. This shows that $\mathcal{S}, s \Vdash \psi_1 \to \psi_2$.

Hence, we have $\mathcal{S}, s_0 \Vdash \Delta$, because $\mathcal{M}, s_0, \rho \Vdash \lceil \Delta \rceil$. Thus $\mathcal{S}, s_0 \Vdash \varphi$. This in turn implies $\mathcal{M}, s_0, \rho \Vdash \lceil \varphi \rceil$. Since $\mathcal{M}$, s_0 and ρ were arbitrary satisfying $\mathcal{M}, s_0, \rho \Vdash \Gamma(\Delta, \varphi), \lceil \Delta \rceil$, we have $\Gamma(\Delta, \varphi), \lceil \Delta \rceil \Vdash_{\mathcal{I}\mathrm{Jp}} \lceil \varphi \rceil$.

Assume $\Gamma(\Delta, \varphi), \lceil \Delta \rceil \vdash_{\mathcal{I}\mathrm{Jp}} \lceil \varphi \rceil$. Let $\mathcal{S}$ be a Kripke NJp-model and $s_0 \in \mathcal{S}$ be such that $\mathcal{S}, s_0 \Vdash \Delta$. We construct a Kripke $\mathcal{I}$Jp-model $\mathcal{M}$ using Theorem 4.2.8. This model has the same states and state ordering as $\mathcal{S}$, and it satisfies the following for every state $s \in \mathcal{S}$ and each $p \in V_p$:

- $\mathcal{M}, s, \rho \Vdash_1 p$ iff $\mathcal{S}, s \Vdash p$,
- $\mathcal{M}, s, \rho \Vdash_0 p$ iff $\mathcal{S}, s \nVdash p$,

where ρ is an $\mathcal{M}$-valuation such that $\rho(p) \equiv \overline{p}$. Thus $\mathcal{M}, \rho \Vdash_1 \mathsf{H}p$ for each $p \in V_p$. Using this it is easy to show by induction on the structure of a formula ψ that $\mathcal{M}, \rho \Vdash_1 \mathsf{H}\lceil\psi\rceil$. It is then straightforward to prove by induction on the structure of ψ that: $\mathcal{M}, s, \rho \Vdash_1 \lceil\psi\rceil$ iff $\mathcal{S}, s \Vdash \psi$.

Hence, we have $\mathcal{M}, s_0, \rho \Vdash \lceil\Delta\rceil$, because $\mathcal{S}, s_0 \Vdash \Delta$. Since $\mathcal{M}, \rho \Vdash \mathsf{H}p$ for each $p \in V_p$, we also have $\mathcal{M}, \rho \Vdash \Gamma(\Delta, \varphi)$. Thus $\mathcal{M}, \rho \Vdash \lceil\varphi\rceil$. So $\mathcal{S} \Vdash \varphi$. Since $\mathcal{S}$ and s_0 were arbitrary satisfying $\mathcal{S}, s_0 \Vdash \Delta$, we obtain $\Delta \Vdash_{\mathrm{NJp}} \varphi$. □

Corollary 4.3.4. $\Delta \vdash_{\mathrm{NJp}} \varphi$ *iff* $\Gamma(\Delta, \varphi), \lceil\Delta\rceil \vdash_{\mathcal{I}\mathrm{Jp}} \lceil\varphi\rceil$.

Proof. Follows from Theorem 4.3.3, Theorem 4.1.8, Theorem 4.1.11 and Theorem 2.4.3. □

Theorem 4.3.5 (Completeness of the translation for $\mathcal{I}$Kp)**.**
$\Delta \models_{\mathrm{NKp}} \varphi$ *iff* $\Gamma(\Delta, \varphi), \lceil\Delta\rceil \models_{\mathcal{I}\mathrm{Kp}} \lceil\varphi\rceil$.

Proof. The proof is similar to the proof of Theorem 4.3.3, but somewhat simpler. First, assume $\Delta \models_{\mathrm{NKp}} \varphi$. Let $\mathcal{M}$ be an $\mathcal{I}$Kp-model and ρ an $\mathcal{M}$-valuation such that $\mathcal{M}, \rho \models \Gamma(\Delta, \varphi), \lceil\Delta\rceil$. Let v be an NKp-valuation defined by:

- $v(p) = 1$ iff $\mathcal{M}, \rho \models p$,
- $v(p) = 0$ iff $\mathcal{M}, \rho \not\models p$.

Using Lemma 4.3.2 and Theorem 4.1.14 it is easy to show by induction on the structure of a subformula ψ of a formula from $\Delta \cup \{\varphi\}$ that: $\mathcal{M}, \rho \models \lceil\psi\rceil$ iff $v \models \psi$. Hence $v \models \Delta$, because $\mathcal{M}, \rho \models \lceil\Delta\rceil$. So $v \models \varphi$. Thus $\mathcal{M}, \rho \models \lceil\varphi\rceil$. Therefore, $\Gamma(\Delta, \varphi), \lceil\Delta\rceil \models_{\mathcal{I}\mathrm{Kp}} \lceil\varphi\rceil$.

In the other direction, assume $\Gamma(\Delta, \varphi), \lceil\Delta\rceil \models_{\mathcal{I}\mathrm{Kp}} \lceil\varphi\rceil$. Let v be an NKp-valuation such that $v \models \Delta$. Take $\mathcal{M}$ to be the $\mathcal{I}$Kp-model obtained by applying Theorem 4.2.13 to v. It is easy to check by induction on the structure of a formula ψ that:

- $\mathcal{M}, \rho \models \mathsf{H}\lceil\psi\rceil$, and
- $\mathcal{M}, \rho \models \lceil\psi\rceil$ iff $v \models \psi$,

where ρ is an $\mathcal{M}$-valuation such that $\rho(p) \equiv \overline{p}$ for $p \in V_P$. Then $\mathcal{M}, \rho \models \Gamma(\Delta, \varphi), \lceil\Delta\rceil$. Thus $\mathcal{M}, \rho \models \lceil\varphi\rceil$. Hence $v \models \varphi$. Therefore, $\Delta \models_{\mathrm{NKp}} \varphi$. □

Corollary 4.3.6. $\Delta \vdash_{\mathrm{NJp}} \varphi$ *iff* $\Gamma(\Delta, \varphi), \lceil\Delta\rceil \vdash_{\mathcal{I}\mathrm{Jp}} \lceil\varphi\rceil$.

Proof. Follows from Theorem 4.3.3, Theorem 4.1.14, Theorem 4.1.16 and Theorem 2.4.5. □

Chapter 5

First-order predicate logic

5.1 Illative systems

Definition 5.1.1. The system $\mathcal{I}$J of *intuitionistic first-order illative combinatory logic* comes in three variants: $\mathcal{I}\mathrm{J}_{\lambda\beta\eta}$, $\mathcal{I}\mathrm{J}_{\lambda\beta}$, $\mathcal{I}\mathrm{J}_{\mathrm{CL}w}$. They differ chiefly in the underlying reduction system. The set of terms $\mathbb{T}$ is defined separately for each of the variants, as in Definition 4.1.1, basing on a signature Σ containing the following illative constants: Ξ, X, A, P, $\mathsf{\Lambda}$, V, $\bot$. We adopt the same abbreviations as in Definition 4.1.1 plus the following (see also Section 1.1):

- $\mathsf{L} \equiv \lambda x.\Xi xx$,
- $M \circ N \equiv \lambda x.M(Nx)$,
- $\forall x : M \,.\, N \equiv \Xi M(\lambda x.N)$ when $x \notin \mathrm{FV}(M)$,
- $\exists x : M \,.\, N \equiv \mathsf{X}M(\lambda x.N)$ when $x \notin \mathrm{FV}(M)$.

Whenever we write $\exists x : X \,.\, Y$ or $\forall x : X \,.\, Y$ we assume that $x \notin \mathrm{FV}(X)$.

A judgement in $\mathcal{I}$J has the form $\Gamma \vdash X$ where Γ is a finite set of terms and X is a term. We adopt the same conventions concerning $\vdash$ as in Definition 4.1.1. The rules of $\mathcal{I}$J consist of the rules of $\mathcal{I}$Jp in Figure 4.1, the rules for quantifiers in Figure 5.1, and the rule (ALI):

$$\frac{}{\Gamma \vdash \mathsf{LA}}\ (\mathsf{ALI})$$

The system $\mathcal{I}$K of *classical first-order illative combinatory logic* is obtained from $\mathcal{I}$J by adding the rule of excluded middle (EM) (see Definition 4.1.1).

Intuitively, $\mathsf{L}X$ means "X is a type" or "X represents a permissible range of quantification". The illative constant A represents a first-order universe – the type of all individuals. We could easily add more such constants to effectively obtain a many-sorted system, but we will not do so to keep things simple. See also Section 1.1.

Informally, the interpretation of quantifiers is as follows:

- ΞXY is true iff X is a type, and for all Z such that XZ is true, YZ is true,

$$\frac{\Gamma, Xx \vdash Yx \quad \Gamma \vdash \mathsf{L}X \quad x \notin \mathrm{FV}(\Gamma, X, Y)}{\Gamma \vdash \Xi XY} \ (\Xi\mathrm{I}) \qquad \frac{\Gamma \vdash \Xi XY \quad \Gamma \vdash XZ}{\Gamma \vdash YZ} \ (\Xi\mathrm{E})$$

$$\frac{\Gamma, Xx \vdash \mathsf{H}(Yx) \quad \Gamma \vdash \mathsf{L}X \quad x \notin \mathrm{FV}(\Gamma, X, Y)}{\Gamma \vdash \mathsf{H}(\Xi XY)} \ (\Xi\mathsf{H}\mathrm{I}) \qquad \frac{\Gamma \vdash \mathsf{H}(\Xi XY)}{\Gamma \vdash \Xi X(\mathsf{H} \circ Y)} \ (\Xi\mathsf{H}\mathrm{E})$$

$$\frac{\Gamma \vdash \mathsf{H}(\Xi XY)}{\Gamma \vdash \mathsf{L}X} \ (\Xi\mathsf{L}\mathrm{E})$$

$$\frac{\Gamma \vdash MZ \quad \Gamma \vdash NZ \quad \Gamma \vdash \mathsf{L}M}{\Gamma \vdash \mathsf{X}MN} \ (\mathsf{XI})$$

$$\frac{\Gamma \vdash \mathsf{X}MN \quad \Gamma, Mx, Nx \vdash Z \quad x \notin \mathrm{FV}(\Gamma, M, N, Z)}{\Gamma \vdash Z} \ (\mathsf{X}\mathrm{E})$$

$$\frac{\Gamma, Mx \vdash \mathsf{H}(Nx) \quad \Gamma \vdash \mathsf{L}M \quad x \notin \mathrm{FV}(\Gamma, M, N)}{\Gamma \vdash \mathsf{H}(\mathsf{X}MN)} \ (\mathsf{XHI})$$

$$\frac{\Gamma \vdash \mathsf{H}(\mathsf{X}MN) \quad \Gamma, \mathsf{X}MN \vdash Z \quad \Gamma, \Xi M(\mathsf{H} \circ N) \vdash Z}{\Gamma \vdash Z} \ (\mathsf{XH}\mathrm{E})$$

$$\frac{\Gamma \vdash \mathsf{H}(\mathsf{X}MN)}{\Gamma \vdash \mathsf{L}M} \ (\mathsf{XL}\mathrm{E})$$

Figure 5.1: Rules for quantifiers (Ξ and X)

- ΞXY is false iff X is a type, and there is Z such that XZ is true and YZ is false, *and* for all Z such that XZ is true, YZ is true or false,
- $\mathsf{X}MN$ is true iff M is a type, and there is Z such that MZ is true and NZ is true,
- $\mathsf{X}MN$ is false iff M is a type, and for all Z such that MZ is true, NZ is false.

Note there is a certain asymmetry in the rules for Ξ and X. It is not true that $\Gamma \vdash_{\mathcal{I}\mathrm{K}} \forall x : X . Y$ is equivalent to $\Gamma \vdash_{\mathcal{I}\mathrm{K}} \neg\exists x : X . \neg Y$.[1] In the classical setting it may be more convenient to simply define $\forall x : X . Y$ as $\neg\exists x : X . \neg Y$. However, this obviously does not work for the intutitionistic system. We do not know how to formulate rules for Ξ in a way that would be satisfactorily simple, would make sense in intuitionistic logic, and after the addition of the rule of excluded middle (EM) would yield the desired equivalence. We shall thus stick with the present formulation of the rules for Ξ.

Systems of illative combinatory logic known to the author do not have the H-elimination rules ($\Xi\mathsf{H}$E) and (XHE), nor the L-elimination rules ($\Xi\mathsf{L}$E) and (XLE). The reasons for including these rules are as with the H-elimination rules for other connectives: they make our semantics complete and are useful in practice.

We could simplify the rules for X by dropping (XLE) and replacing (XI) with

$$\frac{\Gamma \vdash MZ \quad \Gamma \vdash NZ}{\Gamma \vdash \mathsf{X}MN}\ (\mathsf{XI}')$$

It is easy to change the proofs and definitions that follow[2] to work with this modification of our systems. However, this would require complicating the semantics slightly. Condition 13 in Definition 5.1.4 would have to be split into two conditions. We shall thus continue with our original formulation.

Now we consider the question of whether it is possible to define some connectives from the other ones in a way that would make the relevant rules derivable. Certainly, one may define P in terms of Ξ by $\mathsf{P}XY \equiv \Xi(\mathsf{K}X)(\mathsf{K}Y)$. This is a standard definition, but to make all rules for P derivable one would need to add to $\mathcal{I}$J the following rule for Ξ, which intuitively says that if YZ holds for an arbitrary object Z, then ΞXY holds, regardless of what X is (it may not represent a type at all).

$$\frac{\Gamma \vdash Yx \quad x \notin \mathrm{FV}(\Gamma, X, Y)}{\Gamma \vdash \Xi XY}\ (\Xi\mathrm{I}_r)$$

However, such a definition complicates slightly the model constructions, so we will not adopt it. As for the other connectives, it is an open question whether they may be defined from Ξ, or possibly some more constants. Note that the presence of unrestricted λ-abstraction and illative primitives like H provides additional possibilities for such definitions, as compared to

[1] However, as we shall see, in $\mathcal{I}$K all classical tautologies are provable if we restrict the right-hand sides of judgements to terms which are propositions, i.e., terms M such that $\mathsf{H}M$ is provable.

[2] This remark pertains only to the proofs and definitions in the present chapter – for the first-order system. For a higher-order system in the next chapter it is an open problem whether an analogous modification can be made.

ordinary logic. See e.g. [Bun84] for a definition of $\mathsf{\Lambda}$ from Ξ and H which gives unrestricted introduction and elimination rules, though in a somewhat different system, with H essentially being a type, among other differences.

Lemma 5.1.2. *The rules* (Weak), (Sub), (EqL) *and* (Cut) *from Lemma 4.1.2 are admissible in* $\mathcal{I}$J.

Proof. Analogous to Lemma 4.1.2. □

5.1.1 Kripke semantics

In this section we define Kripke semantics for $\mathcal{I}$J.

Definition 5.1.3. A *first-order illative combinatory algebra* (FOICA) is a propositional illative combinatory algebra (see Definition 4.1.4) with additional distinguished elements: A, L, Ξ and x.

Definition 5.1.4. A *Kripke* $\mathcal{I}$J*-model* ($\mathcal{I}\mathrm{J}_{\lambda\beta\eta}$-model, $\mathcal{I}\mathrm{J}_{\lambda\beta}$-model or $\mathcal{I}\mathrm{J}_{\mathrm{CL}w}$-model) is a tuple $\mathcal{S} = \langle \mathcal{C}, I, S, \leq, \sigma_0, \sigma_1 \rangle$ where:

- $\mathcal{C}$ is a first-order illative combinatory algebra (extensional for $\mathcal{I}\mathrm{J}_{\lambda\beta\eta}$, λ-model for $\mathcal{I}\mathrm{J}_{\lambda\beta}$) satisfying $\mathsf{h} \cdot a = \mathsf{p} \cdot a \cdot a$ and $\mathsf{L} \cdot a = \Xi \cdot a \cdot a$ for any $a \in \mathcal{C}$,
- S is a set of states,
- $\leq$ is a partial order on states,
- σ_0 and σ_1 are mappings from $\mathcal{C}$ to S satisfying conditions 1-9 from Definition 4.1.5 and the following:
  10. $s \in \sigma_1(\Xi \cdot a \cdot b)$ iff
      - $s \in \sigma_1(\mathsf{L} \cdot a)$, and
      - for every $s' \geq s$ and $c \in \mathcal{C}$ such that $s' \in \sigma_1(a \cdot c)$ we have $s' \in \sigma_1(b \cdot c)$,
  11. $s \in \sigma_0(\Xi \cdot a \cdot b)$ iff
      - $s \in \sigma_1(\mathsf{L} \cdot a)$, and
      - for every $s' \geq s$ and $c \in \mathcal{C}$ such that $s' \in \sigma_1(a \cdot c)$ we have $s' \in \sigma_h(b \cdot c)$, and
      - there exists $s' \geq s$ and $c \in \mathcal{C}$ such that $s' \in \sigma_1(a \cdot c)$ and $s' \in \sigma_0(b \cdot c)$,
  12. $s \in \sigma_1(\mathsf{x} \cdot a \cdot b)$ iff
      - $s \in \sigma_1(\mathsf{L} \cdot a)$, and
      - there exists $c \in \mathcal{C}$ such that $s \in \sigma_1(a \cdot c)$ and $s \in \sigma_1(b \cdot c)$,
  13. $s \in \sigma_0(\mathsf{x} \cdot a \cdot b)$ iff
      - $s \in \sigma_1(\mathsf{L} \cdot a)$, and
      - for every $s' \geq s$ and $c \in \mathcal{C}$ such that $s' \in \sigma_1(a \cdot c)$ we have $s' \in \sigma_h(b \cdot c)$, and
      - for every $c \in \mathcal{C}$ such that $s \in \sigma_1(a \cdot c)$ we have $s \in \sigma_0(b \cdot c)$,
  14. $\sigma_1(\mathsf{L} \cdot \mathsf{A}) = \mathcal{S}$.

Given an $\mathcal{S}$-valuation $\rho : V \to \mathcal{C}$, the *value* of $M \in \mathbb{T}$, denoted $[\![M]\!]^{\mathcal{S}}_{\rho}$ or just $[\![M]\!]_{\rho}$, is defined analogously to the corresponding notion in Definition 4.1.5, with additional conditions for the new illative constants.

We adopt conventions analogous to those in Definition 4.1.5. In particular, $\Gamma \Vdash_i M$ means that for every $\mathcal{S}$, every $s \in \mathcal{S}$ and every ρ, the condition $s, \rho \Vdash_i \Gamma$ implies $s, \rho \Vdash_i M$. And $\Gamma \Vdash_{\mathcal{I}\mathrm{J}} M$ stands for $\Gamma \Vdash_1 M$.

Note that any Kripke $\mathcal{I}$J-model is a Kripke $\mathcal{I}$Jp-model. It is thus clear that Lemma 4.1.6 also holds for Kripke $\mathcal{I}$J-models. The intuitive interpretation of σ_0, σ_1, σ_h and $\mathcal{S}$ is as for Kripke $\mathcal{I}$Jp-models. See the discussion just after Definition 4.1.5. The statement $s \in \sigma_1(\mathsf{L} \cdot a)$ is intuitively interpreted as "a is a type in state s" or "a determines a permissible range of quantification in state s".

Lemma 5.1.5. *If $\rho' = \rho[x/[\![X]\!]_{\rho}]$ then $[\![Y]\!]_{\rho'} = [\![Y[x/X]]\!]_{\rho}$.*

Proof. If $\mathbb{T} = \mathbb{T}_{\mathrm{CL}}$ then we proceed by induction on the structure of Y. Otherwise, $\mathbb{T} = \mathbb{T}_{\lambda}$ and $\rho' = \rho[x/[\![(X)_{\mathrm{CL}}]\!]_{\rho}]$ and $[\![Y]\!]_{\rho'} = [\![(Y)_{\mathrm{CL}}]\!]_{\rho'}$, so by the case $\mathbb{T} = \mathbb{T}_{\mathrm{CL}}$ we have

$$[\![Y]\!]_{\rho'} = [\![(Y)_{\mathrm{CL}}[x/(X)_{\mathrm{CL}}]]\!]_{\rho}.$$

Thus $[\![Y]\!]_{\rho'} = [\![(Y[x/X])_{\mathrm{CL}}]\!]_{\rho} = [\![Y[x/X]]\!]_{\rho}$ by Lemma 2.3.13. □

For convenience of reference, we now reformulate in terms of $\Vdash_1$ and $\Vdash_0$ the conditions from Definition 5.1.4.

Lemma 5.1.6. *For any Kripke $\mathcal{I}$J-model $\mathcal{S}$ and any valuation ρ the following hold for $s \in \mathcal{S}$, $X, Y \in \mathbb{T}$ and $x \notin \mathrm{FV}(X, Y, Z)$:*

10. $s, \rho \Vdash_1 \Xi XY$ *iff*

- $s, \rho \Vdash_1 \mathsf{L}X$, *and*
- *for every $s' \geq s$ and $c \in \mathcal{C}$ such that $s', \rho[x/c] \Vdash_1 Xx$ we have $s', \rho[x/c] \Vdash_1 Yx$,*

11. $s, \rho \Vdash_0 \Xi XY$ *iff*

- $s, \rho \Vdash_1 \mathsf{L}X$, *and*
- *for every $s' \geq s$ and $c \in \mathcal{C}$ such that $s', \rho[x/c] \Vdash_1 Xx$ we have $s', \rho[x/c] \Vdash_1 \mathsf{H}(Yx)$, and*
- *there exists $s' \geq s$ and $c \in \mathcal{C}$ such that $s', \rho[x/c] \Vdash_1 Xx$ and $s', \rho[x/c] \Vdash_0 Yx$,*

12. $s, \rho \Vdash_1 \mathsf{X}YZ$ *iff*

- $s, \rho \Vdash_1 \mathsf{L}Y$, *and*
- *there exists $c \in \mathcal{C}$ such that $s, \rho[x/c] \Vdash_1 Yx$ and $s, \rho[x/c] \Vdash_1 Zx$,*

13. $s, \rho \Vdash_0 \mathsf{X}YZ$ *iff*

- $s, \rho \Vdash_1 \mathsf{L}Y$, *and*

- *for every* $s' \geq s$ *and* $c \in \mathcal{C}$ *such that* $s', \rho[x/c] \Vdash_1 Yx$ *we have* $s', \rho[x/c] \Vdash_1 \mathsf{H}(Zx)$*, and*
- *for every* $c \in \mathcal{C}$ *such that* $s, \rho[x/c] \Vdash_1 Yx$ *we have* $s, \rho[x/c] \Vdash_0 Zx$*,*

14. $s, \rho \Vdash_1 \mathsf{LA}$ *for any* $s \in \mathcal{S}$*.*

Proof. Follows from definitions. □

Theorem 5.1.7 (Soundness of Kripke semantics for $\mathcal{I}$J)**.**
If $\Gamma \vdash_{\mathcal{I}\mathrm{J}} M$ *then* $\Gamma \Vdash_{\mathcal{I}\mathrm{J}} M$*.*

Proof. The proof is by straightforward induction on the length of derivation of $\Gamma \vdash M$.

Assume $\mathcal{S}$ is a Kripke $\mathcal{I}$J-model, ρ a valuation and $s \in \mathcal{S}$. Suppose $\mathcal{S}, s \Vdash_1 \Gamma$ and consider the last rule used in the derivation of $\Gamma \vdash M$. Since $\mathcal{S}$ is also a Kripke $\mathcal{I}$Jp-model, all rules of $\mathcal{I}$Jp have already been verified in the proof of Theorem 4.1.8. Hence, it remains to consider the following possibilities.

(ALI) Follows directly from condition 14 in Definition 5.1.4.

(ΞI) Then $M \equiv \Xi XY$ and $x \notin \mathrm{FV}(\Gamma, X, Y)$ and $\Gamma, Xx \vdash Yx$ and $\Gamma \vdash \mathsf{L}X$. Let $s' \geq s$ and $c \in \mathcal{C}$ be such that $s', \rho[x/c] \Vdash_1 Xx$. Since $x \notin \mathrm{FV}(\Gamma)$, we have $s', \rho[x/c] \Vdash_1 \Gamma, Xx$. Hence $s', \rho[x/c] \Vdash_1 Yx$ by the IH. Of course, also $s, \rho \Vdash_1 \mathsf{L}X$, by the IH. Therefore, $s, \rho \Vdash_1 M$.

(ΞE) Then $M \equiv YZ$ and $\Gamma \vdash \Xi XY$ and $\Gamma \vdash XZ$. By IH we obtain $s, \rho \Vdash_1 \Xi XY$ and $s, \rho \Vdash_1 XZ$. Thus $s, \rho[x/[\![Z]\!]_\rho] \Vdash_1 Yx$. By Lemma 5.1.5, $s, \rho \Vdash_1 YZ$.

(ΞHI) Then $M \equiv \mathsf{H}(\Xi XY)$ and $x \notin \mathrm{FV}(\Gamma, X, Y)$ and $\Gamma, Xx \vdash \mathsf{H}(Yx)$ and $\Gamma \vdash \mathsf{L}X$. Let $s' \geq s$ and $c \in \mathcal{C}$ be such that $s', \rho[x/c] \Vdash_1 Xx$. Then $s', \rho[x/c] \Vdash_1 \mathsf{H}(Yx)$ by the IH, so $s', \rho[x/c] \Vdash_1 Yx$ or $s', \rho[x/c] \Vdash_0 Yx$. Also $s, \rho \Vdash_1 \mathsf{L}X$. Thus, if there exists $s' \geq s$ and $c \in \mathcal{C}$ such that $s', \rho[x/c] \Vdash_1 Xx$ and $s', \rho[x/c] \Vdash_0 Yx$, then $s, \rho \Vdash_0 \Xi XY$. Otherwise $s, \rho \Vdash_1 \Xi XY$. In any case $s, \rho \Vdash_1 M$.

(ΞHE) Then $M \equiv \Xi X(\mathsf{H} \circ Y)$ and $\Gamma \vdash \mathsf{H}(\Xi XY)$. Let $s' \geq s$ and $c \in \mathcal{C}$ be such that $s', \rho[x/c] \Vdash_1 Xx$. By the IH, $s, \rho \Vdash_1 \Xi XY$ or $s, \rho \Vdash_0 \Xi XY$. In any case, $s', \rho[x/c] \Vdash_1 Xx$ implies $s', \rho[x/c] \Vdash_1 \mathsf{H}(Yx)$. Obviously, also $s, \rho \Vdash_1 \mathsf{L}X$. Thus $s, \rho \Vdash_1 M$.

(ΞLE) Follows from the IH and conditions 10 and 11 in Definition 5.1.4.

(XI) Follows from the IH, condition 12 in Definition 5.1.4, and from Lemma 5.1.5.

(XE) Follows from the IH and condition 12 in Definition 5.1.4.

(XHI) Follows from the IH and conditions 12 and 13 in Definition 5.1.4.

(XHE) Follows from the IH and conditions 4, 10, 12 and 13 in Definition 5.1.4.

(XLE) Follows from the IH and conditions 12 and 13 in Definition 5.1.4.

□

We shall now prove that the semantics based on Kripke $\mathcal{I}$J-models is also complete for $\mathcal{I}$J. For this purpose, we need to augment the definition of primeness.

Definition 5.1.8. A set of terms Γ over a signature Σ is *prime* with respect to Σ if:

- $\Gamma \vdash X$ implies $X \in \Gamma$,
- $\Gamma \vdash X \vee Y$ implies $\Gamma \vdash X$ or $\Gamma \vdash Y$,
- $\Gamma \vdash \mathsf{X}YZ$ implies $\Gamma \vdash Yc \wedge Zc$ for some constant $c \in \Sigma$.

The following simple lemma implies that provability in $\mathcal{I}$J is conservative under extensions of signature.

Lemma 5.1.9. *Let Γ be a set of terms over Σ and M a term over Σ. Let $\Sigma' \supseteq \Sigma$. Let $\vdash_\Sigma$ denote provability in $\mathcal{I}$J with terms over Σ, and $\vdash_{\Sigma'}$ provability in $\mathcal{I}$J with terms over Σ'. Then we have the following equivalence:*

- *$\Gamma \vdash_\Sigma M$ iff $\Gamma \vdash_{\Sigma'} M$.*

Proof. The implication from left to right is obvious. The other direction is proven by induction on the length of derivation, showing that the constants from $\Sigma' \setminus \Sigma$ may be replaced with fresh free variables. □

Lemma 5.1.10. *Assume Σ is a countable signature and C is a countably infinite set of constants, disjoint with Σ, such that $\Sigma \cup C \subseteq \Sigma'$. Let Γ be a set of terms over Σ, and M a term over Σ'.*

If $\Gamma \nvdash M$ then there exists a set $\Gamma' \supseteq \Gamma$ of terms over $\Sigma \cup C$, which is prime with respect to $\Sigma \cup C$ and satisfies $\Gamma' \nvdash M$.

Proof. Without loss of generality we may assume that $C = \bigcup_{n \in \mathbb{N}} C_n$ where C_n are pairwise disjoint countably infinite sets of constants. Because the number of constants occuring in M is finite, we may also assume that none of the constants in C occur in M.

We define by induction sets Γ_n of terms over signature $\Sigma_n = \Sigma \cup \bigcup_{k \le n} C_k$ such that $\Gamma_n \nvdash M$. We take $\Gamma_0 = \Gamma$. Now suppose we have defined Γ_n. Since Σ_n, and thus the set of terms over Σ_n, is countable, we may assume that C_{n+1} contains a distinct constant c_ξ for each term $\xi \equiv \mathsf{X}YZ$ in Γ_n. Consider the set $\mathcal{X}$, ordered by inclusion, of all $\mathcal{A} \subseteq \mathbb{T}(\Sigma_{n+1})$ such that:

(a) $\mathcal{A} \supseteq \Gamma_n$,

(b) $\mathcal{A} \nvdash M$,

(c) if $\xi \equiv \mathsf{X}YZ$ is in Γ_n and c_ξ occurs in some term in $\mathcal{A}$, then $Yc_\xi \wedge Zc_\xi \in \mathcal{A}$.

It is easy to see that every non-empty chain L of elements of $\mathcal{X}$ has an upper bound $\bigcup L \in \mathcal{X}$. Of course, also $\mathcal{X} \neq \emptyset$, because $\Gamma_n \in \mathcal{X}$ (note that the second condition follows from the inductive hypothesis). Therefore, by Zorn's Lemma, there exists a maximal element in $\mathcal{X}$, and we take Γ_{n+1} to be any such maximal element. Obviously, we then have $\Gamma_{n+1} \nvdash M$.

We prove the following:

1. if $\xi \equiv \mathsf{X}YZ$ is in Γ_n then c_ξ occurs in some term in Γ_{n+1} (and thus $Yc_\xi \wedge Zc_\xi \in \Gamma_{n+1}$ by (c)),

2. if $\Gamma_{n+1} \vdash X$ then $X \in \Gamma_{n+1}$,
3. if $\Gamma_{n+1} \vdash X \vee Y$ then $\Gamma_{n+1} \vdash X$ or $\Gamma_{n+1} \vdash Y$.

Suppose c_ξ does not occur in any term in Γ_{n+1}. We show that $\Gamma_{n+1} \cup \{N\}$ contradicts the maximality of Γ_{n+1}, where $N \equiv Yc_\xi \wedge Zc_\xi$. It suffices to show that $\Gamma_{n+1}, N \nvdash M$. Assume otherwise. Since c_ξ is a constant not occuring in any term in Γ_{n+1} or in M, we may as well change it to a fresh variable x. Thus we have $\Gamma_{n+1}, Yx \wedge Zx \vdash M$. It is easy to see that then $\Gamma_{n+1}, Yx, Zx \vdash M$. Since $\xi \in \Gamma_n \subseteq \Gamma_{n+1}$, we have $\Gamma_{n+1} \vdash \xi$. So by (XE) we obtain $\Gamma_{n+1} \vdash M$. Contradiction.

Now let $\mathcal{A}$ be any superset of Γ_{n+1}. Of course $\mathcal{A} \supseteq \Gamma_n$, because $\Gamma_n \subseteq \Gamma_{n+1}$. Using the implication just proven, it is easy to see that $\mathcal{A}$ also satisfies (c).

Suppose $\Gamma_{n+1} \vdash X$ and $X \notin \Gamma_{n+1}$. We show that $\Gamma_{n+1} \cup \{X\} \in \mathcal{X}$, which contradicts the maximality of Γ_{n+1}. It suffices to show (b), since the conditions (a) and (c) were shown in the previous paragraph. If $\Gamma_{n+1}, X \vdash M$ and $\Gamma_{n+1} \vdash X$, then $\Gamma_{n+1} \vdash M$ by (Cut), contradiction. So $\Gamma_{n+1}, X \nvdash M$ and thus (b) is satisfied for $\Gamma_{n+1} \cup \{X\}$.

Suppose $\Gamma_{n+1} \vdash X \vee Y$ and $\Gamma_{n+1} \nvdash X$ and $\Gamma_{n+1} \nvdash Y$. Then either $\Gamma_{n+1} \cup \{X\}$ or $\Gamma_{n+1} \cup \{Y\}$ contradicts the maximality of Γ_{n+1}. It suffices to show that at least one of $\Gamma_{n+1} \cup \{X\}$ or $\Gamma_{n+1} \cup \{Y\}$ satisfies (b). Assuming otherwise, $\Gamma_{n+1}, X \vdash M$ and $\Gamma_{n+1}, Y \vdash M$, so $\Gamma_{n+1} \vdash M$ – contradiction.

Now, we finally take $\Sigma' = \bigcup_{n \in \mathbb{N}} \Sigma_n$ and $\Gamma' = \bigcup_{n \in \mathbb{N}} \Gamma_n$. It is easy to see that Γ' is prime and $\Gamma' \nvdash M$. This follows from the fact that if $\Gamma' \vdash X$ then $\mathcal{A} \vdash X$ for some finite subset $\mathcal{A} \subseteq \Gamma'$. So there exists n such that $\mathcal{A} \subseteq \Gamma_n$ and $X \in \mathbb{T}(\Sigma_n)$. Thus $\Gamma_n \vdash X$, so $X \in \Gamma_{n+1}$, from which the claim easily follows using the three implications shown above. □

The proof of completeness for $\mathcal{I}$J is similar to the proof of Theorem 4.1.11.

Theorem 5.1.11 (Completeness of Kripke semantics for $\mathcal{I}$J)**.**
If $\Gamma \Vdash_{\mathcal{I}\mathrm{J}} M$ then $\Gamma \vdash_{\mathcal{I}\mathrm{J}} M$.

Proof. Assume $\Gamma \nvdash M$. We construct a Kripke $\mathcal{I}$J-model $\mathcal{S} = \langle \mathcal{C}, I, S, \leq, \sigma_0, \sigma_1 \rangle$ and a valuation ρ such that there exists a state $s \in \mathcal{S}$ with $s, \rho \Vdash \Gamma$ but $s, \rho \nVdash M$.

Let $C_1 \subseteq C_2 \subseteq \ldots$ be countable sets of constants disjoint with Σ and such that $C_{n+1} \setminus C_n$ is infinite for each $n \in \mathbb{N}$. Let $\Sigma_n = \Sigma \cup C_n$, $C = \bigcup_{n \in \mathbb{N}} C_n$ and $\Sigma' = \Sigma \cup C$.

As the carrier of $\mathcal{C}$ we take $\beta\eta$-equality (for $\mathcal{I}\mathrm{J}_{\lambda\beta}$: β-equality, for $\mathcal{I}\mathrm{J}_{\mathrm{CL}w}$: w-equality) equivalence classes of terms from $\mathbb{T}(\Sigma')$. We will denote by $[X]$ the equivalence class of X. We take $\mathsf{k} = [\mathsf{K}]$, $\mathsf{s} = [\mathsf{S}]$, etc. As h we take $[\lambda x.\mathsf{P}xx]$, and as L we take $[\lambda x.\Xi xx]$. Application is defined by $[X] \cdot [Y] = [XY]$. It is easy to check that $\mathcal{C}$ is a FOICA which is extensional (for $\mathcal{I}\mathrm{Jp}_{\lambda\beta}$: a λ-model) and it satisfies $\mathsf{h} \cdot a = \mathsf{p} \cdot a \cdot a$ and $\mathsf{L} \cdot a = \Xi \cdot a \cdot a$ for any $a \in \mathcal{C}$.

The set of states S is defined as the set of all pairs $\langle \Gamma', \Sigma_n \rangle$ such that Γ' is a consistent set of terms over Σ_n which is prime with respect to Σ_n. Because $\Gamma \nvdash M$, the set S is non-empty, by Lemma 5.1.10. We define:

- $\sigma_1([X]) = \{\langle \Gamma', \Sigma_n \rangle \in S \mid \Gamma' \vdash X\}$,
- $\sigma_0([X]) = \{\langle \Gamma', \Sigma_n \rangle \in S \mid \Gamma' \vdash \mathsf{H}X \text{ and } \Gamma' \nvdash X\}$.

Note that σ_0 and σ_1 are well-defined because of rule (Eq). The order $\leq$ on states is defined as follows: $\langle \Gamma_1, \Sigma_n \rangle \leq \langle \Gamma_2, \Sigma_m \rangle$ iff $\Gamma_1 \subseteq \Gamma_2$ and $n \leq m$.

It remains to check that the conditions on σ_0 and σ_1 from Definition 5.1.4 are satisfied. Conditions 1-9 follow by proofs analogous to those in the proof of Theorem 4.1.11, but using Lemma 5.1.10 instead of Lemma 4.1.10. We check the remaining conditions.

10. The implication from left to right follows from rules (HI), ($\Xi\mathsf{LE}$) and (ΞE). For the other direction, suppose Γ' is a consistent set of terms over Σ_n which is prime with respect to Σ_n, and $\Gamma' \vdash \mathsf{L}X$, and:

    ($\star$) for all $\langle \Gamma'', \Sigma_m \rangle \geq \langle \Gamma', \Sigma_n \rangle$ and all Z such that $\Gamma'' \vdash XZ$ we have $\Gamma'' \vdash YZ$.

    Let $x \notin \mathrm{FV}(\Gamma', X, Y)$ be a fresh variable, and assume $\Gamma', Xx \nvdash Yx$. Let $k \geq n$ be such that $X \in \mathbb{T}(\Sigma_k)$. Then by Lemma 5.1.10 there exists a set $\Gamma'' \supseteq \Gamma' \cup \{Xx\}$ of terms over Σ_{k+1} which is prime with respect to Σ_{k+1} and satisfies $\Gamma'' \nvdash Yx$. But this contradicts ($\star$). Therefore, $\Gamma', Xx \vdash Yx$, and since also $\Gamma' \vdash \mathsf{L}X$, we obtain $\Gamma' \vdash \Xi XY$ by rule (ΞI).

11. The implication from left to right follows from rules ($\Xi\mathsf{LE}$), ($\Xi\mathsf{HE}$) and (Ξ), and from Lemma 5.1.10. For the other direction, suppose Γ' is a consistent set of terms over Σ_n which is prime with respect to Σ_n, and

    - $\Gamma' \vdash \mathsf{L}X$,
    - for every $\langle \Gamma'', \Sigma_m \rangle \geq \langle \Gamma', \Sigma_n \rangle$ and every Z such that $\Gamma'' \vdash XZ$ we have $\Gamma'' \vdash \mathsf{H}(YZ)$,
    - there is $\langle \Gamma_0, \Sigma_{n_0} \rangle \geq \langle \Gamma', \Sigma_n \rangle$ and Z_0 such that $\Gamma_0 \vdash XZ_0$ but $\Gamma_0 \nvdash YZ_0$.

    Using Lemma 5.1.10 and rule ($\Xi\mathsf{HI}$) we may show $\Gamma' \vdash \mathsf{H}(\Xi XY)$, by a proof analogous to the proof of $\Gamma' \vdash \Xi XY$ in the previous point. Assume also $\Gamma' \vdash \Xi XY$. Then $\Gamma_0 \vdash \Xi XY$, because $\Gamma_0 \supseteq \Gamma'$. Since also $\Gamma_0 \vdash XZ_0$ we obtain $\Gamma_0 \vdash YZ_0$ by rule (ΞE). Contradiction.

12. The implication from left to right follows from rules (HI) and (XLE), and from primeness. The implication from right to left follows from rule (XI).

13. The implication from left to right follows from primeness and rules (XLE), (XHE), (ΞE) and (XI). The implication from right to left follows from rule (XHI), Lemma 5.1.10 and primeness.

14. Follows from (ALI).

We define the valuation ρ by $\rho(x) = [x]$. By Lemma 5.1.10 there exists a set $\Gamma' \supseteq \Gamma$ of terms over Σ_1 which is prime with respect to Σ_1 and satisfies $\Gamma' \nvdash M$. So $\langle \Gamma', \Sigma_1 \rangle \in S$. It is easy to check that $\langle \Gamma', \Sigma_1 \rangle, \rho \Vdash \Gamma$ but $\langle \Gamma', \Sigma_1 \rangle, \rho \nVdash M$. □

5.1.2 Classical semantics

In this section we define two variants of classical semantics for $\mathcal{I}$K. The first one is based on classical $\mathcal{I}$K-models, which are simply single-state Kripke $\mathcal{I}$J-models. In contrast to the propositional case, we have not been able to show that this semantics is complete. There is one subtlety which prevents a straightforward adaptation of the standard Henkin-style completeness proof. In fact, it seems plausible that classical $\mathcal{I}$K-models may not be complete for $\mathcal{I}$K. We will explain this in more detail later.

The second semantics, which is complete, is based on Kripke $\mathcal{I}$K-models, which are Kripke $\mathcal{I}$J-models $\langle \mathcal{C}, I, \mathcal{S}, \leq, \sigma_0, \sigma_1 \rangle$ satisfying: for all $s \in \mathcal{S}$ and $a \in \mathcal{C}$, if $s \in \sigma_h(a)$ then $s \in \sigma_1(\mathsf{v} \cdot a \cdot (\mathsf{p} \cdot a \cdot \bot))$.

Definition 5.1.12. A classical $\mathcal{I}$K-model is a Kripke $\mathcal{I}$J-model with exactly one state s_0. For a classical $\mathcal{I}$K-model we adopt the abbreviations $\mathcal{T} = \{a \mid s_0 \in \sigma_1(a)\}$ and $\mathcal{F} = \{a \mid s_0 \in \sigma_0(a)\}$. Note that a FOICA $\mathcal{C}$ and the sets $\mathcal{T}$ and $\mathcal{F}$ uniquely determine a classical $\mathcal{I}$K-model. We sometimes say that a tuple $\mathcal{M} = \langle \mathcal{C}, I, \mathcal{T}, \mathcal{F} \rangle$ is a classical $\mathcal{I}$K-model.

For convenience of reference, we reformulate in terms of $\mathcal{T}$ and $\mathcal{F}$ the conditions on σ_0 and σ_1 from Definition 5.1.4. The reformulation of conditions 1-9 is as in Definition 4.1.12. The remaining conditions are reformulated as follows:

10. $\Xi \cdot a \cdot b \in \mathcal{T}$ iff
    - $\mathsf{L} \cdot a \in \mathcal{T}$, and
    - for every $c \in \mathcal{C}$ such that $a \cdot c \in \mathcal{T}$ we have $b \cdot c \in \mathcal{T}$,
11. $\Xi \cdot a \cdot b \in \mathcal{F}$ iff
    - $\mathsf{L} \cdot a \in \mathcal{T}$, and
    - for every $c \in \mathcal{C}$ such that $a \cdot c \in \mathcal{T}$ we have $b \cdot c \in \mathcal{T} \cup \mathcal{F}$, and
    - there exists $c \in \mathcal{C}$ such that $a \cdot c \in \mathcal{T}$ and $b \cdot c \in \mathcal{F}$,
12. $\mathsf{X} \cdot a \cdot b \in \mathcal{T}$ iff
    - $\mathsf{L} \cdot a \in \mathcal{T}$, and
    - there exists $c \in \mathcal{C}$ such that $a \cdot c \in \mathcal{T}$ and $b \cdot c \in \mathcal{T}$,
13. $\mathsf{X} \cdot a \cdot b \in \mathcal{F}$ iff
    - $\mathsf{L} \cdot a \in \mathcal{T}$, and
    - for every $c \in \mathcal{C}$ such that $a \cdot c \in \mathcal{T}$ we have $b \cdot c \in \mathcal{F}$,
14. $\mathsf{L} \cdot \mathsf{A} \in \mathcal{T}$.

For a classical $\mathcal{I}$K-model $\mathcal{M}$ and a valuation ρ, the notations $\mathcal{M}, \rho \models_i M$, $\rho \models_i M$, $\mathcal{M}, \rho \models_i \Gamma$, etc., are defined as in Definition 4.1.12.

The intuitive interpretation of $\mathcal{T}$ and $\mathcal{F}$ is the same as for classical $\mathcal{I}$Kp-models: $\mathcal{T}$ is the set of true elements, and $\mathcal{F}$ is the set of false elements.

Theorem 5.1.13 (Soundness of semantics for $\mathcal{I}$K based on classical $\mathcal{I}$K-models)**.**
If $\Gamma \vdash_{\mathcal{I}\mathrm{K}} M$ *then* $\Gamma \models_{\mathcal{I}\mathrm{K}} M$.

Proof. The proof proceeds by induction on the length of derivation of $\Gamma \vdash M$, like in the proof of Theorem 5.1.7. Only the additional rule (EM) needs to be checked, which is done in exactly the same way as in the proof of Theorem 4.1.14. □

As we have remarked, it is an open problem whether the semantics based on classical $\mathcal{I}$K-models is complete for $\mathcal{I}$K. The standard Henkin-style completeness proof cannot be easily adapted. To see where the problem is, consider conditions 13 in Definition 5.1.12. To show this condition for a one-state model we would need to have the following for prime Γ:

($\star$) if $\Gamma \vdash \mathsf{L}X$, and for every Z, $\Gamma \vdash XZ$ implies $\Gamma \vdash \neg Y[x/Z]$, then $\Gamma \vdash \neg(\exists x : X . Y)$.

If $\Gamma \vdash (\exists x : X . Y) \vee \neg(\exists x : X . Y)$ then ($\star$) follows from the primeness of Γ, but we have excluded middle only for terms $\exists x : X . Y$ for which $\mathsf{H}(\exists x : X . Y)$ is provable. Essentially, this makes it impossible to easily adapt the standard trick with Henkin constants in the case we have only one state. This observation also makes it plausible that classical $\mathcal{I}$K-models may in fact not be complete for $\mathcal{I}$K. However, proving this would probably be difficult. One would need to find a term M which is not provable in $\mathcal{I}$K but is true in all classical $\mathcal{I}$K-models. Finding such a term would imply the consistency of $\mathcal{I}$K. Moreover, the model construction for $\mathcal{I}$K that we provide to show consistency is a construction of a classical $\mathcal{I}$K-model, so it cannot be used to settle this question.

Definition 5.1.14. A *Kripke $\mathcal{I}$K-model* is a Kripke $\mathcal{I}$J-model $\langle \mathcal{C}, I, \mathcal{S}, \leq, \sigma_0, \sigma_1 \rangle$ satisfying:

- for all $s \in \mathcal{S}$ and $a \in \mathcal{C}$, if $s \in \sigma_h(a)$ then $s \in \sigma_1(\mathsf{v} \cdot a \cdot (\mathsf{p} \cdot a \cdot \bot))$.

Semantics based on Kripke $\mathcal{I}$K-models seems somewhat less intuitive, but it is easy to see that it is sound and complete for $\mathcal{I}$K. We state the relevant theorems without proofs, since they are straightforward modifications of the proofs for $\mathcal{I}$J. One just needs to consider the additional cases to account for the condition in Definition 5.1.14. We shall denote by $\Vdash_{k\mathcal{I}\mathrm{K}}$ the semantic consequence relation with respect to Kripke $\mathcal{I}$K-models.

Theorem 5.1.15 (Soundness of the semantics for $\mathcal{I}$K based on Kripke $\mathcal{I}$K-models)**.**
If $\Gamma \vdash_{\mathcal{I}\mathrm{K}} M$ *then* $\Gamma \Vdash_{k\mathcal{I}\mathrm{K}} M$*.*

Theorem 5.1.16 (Completeness of the semantics for $\mathcal{I}$K based on Kripke $\mathcal{I}$K-models)**.**
If $\Gamma \Vdash_{k\mathcal{I}\mathrm{K}} M$ *then* $\Gamma \vdash_{\mathcal{I}\mathrm{K}} M$*.*

5.2 Model constructions

In this section we construct models for $\mathcal{I}$J and $\mathcal{I}$K. A corollary is consistency of these systems. Like in Section 4.2, the constructions are parameterised by appropriate models for traditional systems, and used later to show completeness of translations of corresponding traditional systems into $\mathcal{I}$J and $\mathcal{I}$K.

5.2.1 Model construction for $\mathcal{I}$J

Fix a Kripke NJ-model $\mathcal{S} = \langle S, \leq, \{\mathcal{A}_s \mid s \in S\} \rangle$. We construct a Kripke $\mathcal{I}$J-model $\mathcal{M}$ parameterised by $\mathcal{S}$. The construction is a relatively straightforward extension of the construction from Section 4.2.1. We assume that function and relation symbols of NJ are present in the syntax of $\mathcal{I}$J. Also, for the purpose of constructing the model, we assume[3] that each element of $\bigcup_{s\in S}\mathcal{A}_s$ occurs as a distinct constant in the set of terms $\mathbb{T}$. Without loss of generality we may assume that for $s_1, s_2 \in S$, if there is no s_0 with $s_0 \le s_1$ and $s_0 \le s_2$, then $\mathcal{A}_{s_1} \cap \mathcal{A}_{s_2} = \emptyset$. We will construct the model $\mathcal{M}$ from appropriate equivalence classes of terms from $\mathbb{T}$. As in Section 4.2.1, we adopt the abbreviation $\top \equiv \mathsf{P}\bot\bot$. In this section we adopt the convention $\mathsf{L}X \equiv \Xi XX$, i.e., when we write $\mathsf{L}X$ this stands for ΞXX, not for $(\lambda x.\Xi xx)X$. This convention is to shorten notations. The important thing is that $\mathsf{L}X$ is never a redex.

Definition 5.2.1. We define binary relation $\to_R$ on $\mathbb{T}$ as the compatible closure of the following rules:

- rules of β- and η-reduction,
- $fa_1 \ldots a_n \to a$ if f is a function symbol and there is $s \in S$ such that $a_1, \ldots, a_n \in \mathcal{A}_s$ and $f^{\mathcal{A}_s}(a_1, \ldots, a_n) = a$.

Denote by $\to_F$ the compatible closure of the rules for function symbols above. Assume $fa_1 \ldots a_n \to a$ and $fa_1 \ldots a_n \to b$. Then $a = f^{\mathcal{A}_{s_1}}(a_1, \ldots, a_n)$, $b = f^{\mathcal{A}_{s_2}}(a_1, \ldots, a_n)$ and $a_1, \ldots, a_n \in \mathcal{A}_{s_1} \cap \mathcal{A}_{s_2}$ for some $s_1, s_2 \in S$. So there is $s_0 \le s_1, s_2$ with $a_1, \ldots, a_n \in \mathcal{A}_{s_0}$. But this implies $f^{\mathcal{A}_{s_1}}(a_1, \ldots, a_n) = f^{\mathcal{A}_{s_2}}(a_1, \ldots, a_n) = f^{\mathcal{A}_{s_0}}(a_1, \ldots, a_n)$. Hence $\to_F$ is confluent. Using Lemma 2.3.4 one also easily shows that $\to_F$ commutes with $\beta\eta$-reduction. Therefore, it follows from the Hindley-Rosen lemma that $\to_R$ is confluent (see Lemma 2.3.3).

Definition 5.2.2. For $s \in S$ and an ordinal α we inductively define binary relations $\succ_s^\alpha$ on $\mathbb{T}$ by the rules from Definition 4.2.1, except $(V_\top)$ and $(V_\bot)$, plus the following rules, where the relation $\leadsto_s^\alpha$ is given by: $X \leadsto_s^\alpha Y$ iff $X \xrightarrow{*}_R \cdot \succ_s^\alpha Y$. The notations $\leadsto_s^{<\alpha}$ and $\succ_s^{<\alpha}$ are as in Definition 4.2.1.

$(\mathsf{LA}_\top)$ $\mathsf{LA} \succ_s^\alpha \top$,

$(\mathsf{A}_\top)$ $\mathsf{A}a \succ_s^\alpha \top$ if $a \in \mathcal{A}_s$,

$(r_\top)$ $ra_1 \ldots a_n \succ_s^\alpha \top$ if $a_1, \ldots, a_n \in \mathcal{A}_s$ and $r^{\mathcal{A}_s}(a_1, \ldots, a_n)$ holds,

$(r_\bot)$ $ra_1 \ldots a_n \succ_s^\alpha \bot$ if $a_1, \ldots, a_n \in \mathcal{A}_s$ and $r^{\mathcal{A}_s}(a_1, \ldots, a_n)$ does not hold,

$(\Xi\mathsf{A}_\top)$ $\Xi\mathsf{A}X \succ_s^\alpha \top$ if for every $s' \ge s$ and $a \in \mathcal{A}_{s'}$ we have $Xa \leadsto_{s'}^{<\alpha} \top$,

$(\Xi\mathsf{A}_\bot)$ $\Xi\mathsf{A}X \succ_s^\alpha \bot$ if

- for every $s' \ge s$ and $a \in \mathcal{A}_{s'}$ we have $Xa \leadsto_{s'}^{<\alpha} \rho$ with $\rho \in \{\top, \bot\}$,
- there exists $s' \ge s$ and $a \in \mathcal{A}_{s'}$ such that $Xa \leadsto_{s'}^{<\alpha} \bot$,

$(\mathsf{XA}_\top)$ $\mathsf{XA}X \succ_s^\alpha \top$ if there exists $a \in \mathcal{A}_s$ such that $Xa \leadsto_s^{<\alpha} \top$,

$(\mathsf{XA}_\top)$ $\mathsf{XA}X \succ_s^\alpha \bot$ if

[3] This assumption holds only in the present section – in general the syntax of $\mathcal{I}$J is not assumed to be parameterised by any specific Kripke NJ-model. It is more convenient to extend the terms of $\mathcal{I}$J and build the Kripke $\mathcal{I}$J-model from equivalence classes of these terms, then to define a separate class of terms specifically for the model construction.

– for every $s' \geq s$ and $a \in \mathcal{A}_{s'}$ we have $Xa \rightsquigarrow_{s'}^{<\alpha} \rho$ with $\rho \in \{\top, \bot\}$,
– for every $a \in \mathcal{A}_s$ we have $Xa \rightsquigarrow_s^{<\alpha} \bot$.

Above X is an arbitrary term.

Like in Section 4.2.1, we assume that $s \in \mathcal{S}$, f and r are function and relation symbols, $\rho, \rho', \ldots \in \{\top, \bot\}$ and M, N, X, Y, Z, etc., are terms, unless otherwise stated.

Lemma 5.2.3. *If $X \succ_s^\alpha \rho$ and $X \xrightarrow{*}_R Y$, then $Y \succ_s^\alpha \rho$.*

Proof. The proof is completely analogous to the proof of Lemma 4.2.2. One just needs to consider additional easy cases corresponding to new rules in Definition 5.2.2. □

Corollary 5.2.4. *$X \rightsquigarrow_s^\alpha Y$ iff there exists X' such that $X =_R X' \succ_s^\alpha Y$.*

Lemma 5.2.5. *The following conditions hold.*

1. *If $M \succ_s^\alpha \top$ and $s' \geq s$ then $M \succ_{s'}^\alpha \top$.*
2. *If $M \succ_s^\alpha \bot$ and $s' \geq s$ then $M \succ_{s'}^\alpha \top$ or $M \succ_{s'}^\alpha \bot$.*

Proof. The proof is analogous to the proof of Lemma 4.2.4. The additional cases are straightforward. □

Lemma 5.2.6. *The following conditions hold.*

1. *If $M \succ_s^{<\alpha} \rho$ then $M \succ_s^\alpha \rho$.*
2. *If $M \succ_s^\alpha \top$ then $M \not\succ_s^\alpha \bot$.*

Proof. Again, the proof is analogous to the proof of Lemma 4.2.5, the additional cases being straightforward. □

Like in Section 4.2.1, it follows from Lemma 5.2.6 and Theorem 2.1.3 that there exists the closure ordinal ζ, i.e., the least ordinal such that $\succ_s^\zeta = \succ_s^{<\zeta}$ for each $s \in \mathcal{S}$. We write $\succ_s$ and $\rightsquigarrow_s$ without superscripts to denote $\succ_s^\zeta$ and $\rightsquigarrow_s^\zeta$. If the set of states S is finite and for all $s \in S$ the structure $\mathcal{A}_s$ is finite, then $\zeta = \omega$. In general, ζ may depend on the Kripke NJ-model $\mathcal{S}$.

Lemma 5.2.3 and the second part of Lemma 5.2.6 imply the following corollary.

Corollary 5.2.7. *The reduction system $\langle \rightarrow_R, \{\succ_s\}_{s \in \mathcal{S}} \rangle$ is coherent.*

Coherence implies that the following is a good definition.

Definition 5.2.8. Define $\mathcal{M}_\mathcal{S} = \langle \mathcal{C}, I, S, \leq, \sigma_0, \sigma_1 \rangle$ where:

- $\mathcal{C}$ is the extensional first-order illative combinatory algebra constructed from the $\overset{*}{\leftrightarrow}_R$-equivalence classes of terms, with $\mathsf{k} = [\mathsf{K}]$, $\mathsf{s} = [\mathsf{S}]$, $\mathsf{p} = [\mathsf{P}]$, etc., where by $[X]$ we denote the equivalence class of X,
- I is defined by $I(c) = [c]$,

- S is the set of states of $\mathcal{S}$,
- $\leq$ is the order on states from $\mathcal{S}$,
- $\sigma_1([X]) = \{s \in S \mid X \rightsquigarrow_s \top\}$,
- $\sigma_0([X]) = \{s \in S \mid X \rightsquigarrow_s \bot\}$.

Theorem 5.2.9. *The structure $\mathcal{M}_{\mathcal{S}}$ is a Kripke $\mathcal{I}$J-model such that for each relation symbol r (in the signature of* NJ*) there is $\overline{r} \in \mathcal{C}$, and for each function symbol f there is $\overline{f} \in \mathcal{C}$, and for each $s \in S$ and each $a \in \mathcal{A}_s$ there is $\overline{a} \in \mathcal{C}$, so that for $s \in S$ and $a_1, \ldots, a_n, a \in \mathcal{A}_s$:*

- $\overline{f} \cdot \overline{a_1} \cdot \ldots \cdot \overline{a_n} = \overline{a}$ *iff* $f^{\mathcal{A}}(a_1, \ldots, a_n) = a$,
- $s \in \sigma_1(\overline{r} \cdot \overline{a_1} \cdot \ldots \cdot \overline{a_n})$ *iff* $r^{\mathcal{A}_s}(a_1, \ldots, a_n)$ *holds,*
- $s \in \sigma_0(\overline{r} \cdot \overline{a_1} \cdot \ldots \cdot \overline{a_n})$ *iff* $r^{\mathcal{A}_s}(a_1, \ldots, a_n)$ *does not hold.*

Proof. Using Corollary 5.2.7 it is straightforward to check that $\mathcal{M}_{\mathcal{S}}$ is a Kripke $\mathcal{I}$J-model satisfying the required conditions. □

Corollary 5.2.10. *The system $\mathcal{I}$J is consistent, i.e., $\nvdash_{\mathcal{I}\mathrm{J}} \bot$.*

5.2.2 Model construction for $\mathcal{I}$K

The construction for $\mathcal{I}$K is an extension of the construction for $\mathcal{I}$Kp from Section 4.2.2. Let $\mathcal{A}$ be a classical NK-structure. We construct a classical $\mathcal{I}$K-model (Definition 5.1.12). The construction is parameterised by $\mathcal{A}$.

We assume that the function and relation symbols of NK are present in the syntax of $\mathcal{I}$K. For the model construction, we also assume[4] that all elements of $\mathcal{A}$ occur as distinct constants in the set of terms $\mathbb{T}$. As in Section 5.2.1, we adopt the convention $\mathsf{L}X \equiv \Xi XX$.

Definition 5.2.11. We define a reduction system $R = \langle \rightarrow_R, \{\succ\} \rangle$ by the rules for reduction

- rules of η- and β-reduction,
- $fa_1 \ldots a_n \rightarrow_R a$ if $a_1, \ldots, a_n \in \mathcal{A}$ and $f^{\mathcal{A}}(a_1, \ldots, a_n) = a$,

the rules from Definition 4.2.10, except $(V_\top)$ and $(V_\bot)$, and the following rules:

($\mathsf{LA}_\top$) $\mathsf{LA} \succ \top$,

($r_\top$) $ra_1 \ldots a_n \succ \top$ if $a_1, \ldots, a_n \in \mathcal{A}$ and $r^{\mathcal{A}}(a_1, \ldots, a_n)$ holds,

($r_\top$) $ra_1 \ldots a_n \succ \top$ if $a_1, \ldots, a_n \in \mathcal{A}$ and $r^{\mathcal{A}}(a_1, \ldots, a_n)$ does not hold,

($\Xi\mathsf{A}_\top$) $\Xi\mathsf{A}X \succ \top$ if for every $a \in \mathcal{A}$ we have $Xa \rightsquigarrow \top$,

($\Xi\mathsf{A}_\bot$) $\Xi\mathsf{A}X \succ \bot$ if

- for every $a \in \mathcal{A}$ there is $\rho \in \{\top, \bot\}$ with $Xa \rightsquigarrow \rho$,
- there exists $a \in \mathcal{A}$ such that $Xa \rightsquigarrow \bot$,

[4]This assumption holds within the present section. Cf. Section 5.2.1.

($\mathsf{XA}_\top$) $\mathsf{XA}X \succ \top$ if there exists $a \in \mathcal{A}$ such that $Xa \rightsquigarrow \top$,

($\mathsf{XA}_\bot$) $\mathsf{XA}X \succ \bot$ if for every $a \in \mathcal{A}$ we have $Xa \rightsquigarrow \bot$,

where $X \rightsquigarrow Y$ denotes $X \xrightarrow{*} \cdot \succ Y$.

If the structure $\mathcal{A}$ is finite then ω is the closure ordinal of the definition of $\succ$. In general, the closure ordinal depends on $\mathcal{A}$ and it may be quite large even for countable structures. Indeed, we conjecture (but we have not checked the details) that if the structure $\mathcal{A}$ is sufficiently rich (essentially includes the natural numbers with enough operations on them), then the closure ordinal ζ is at least the Church-Kleene ordinal ω_1^{CK}, i.e., the first non-recursive ordinal (see e.g. [Rog67, §11.7-8]). Indeed, if ζ were recursive then, by encoding ordinals below ζ with natural numbers, we could essentially replicate the definition of $\rightsquigarrow$ inside the structure $\mathcal{M}_\mathcal{A}$ (see Definition 5.2.13 below), i.e., we could define a term T such that $TX \rightsquigarrow \top$ iff X is the code of a true element of $\mathcal{M}_\mathcal{A}$. By a diagonal argument this would lead to a contradiction.

Lemma 5.2.12. *The reduction system R is coherent.*

Proof. We check the conditions in the definition of coherence. The compatible closure of the reduction rules for function symbols is a confluent relation. Using Lemma 2.3.4 one also easily checks that reduction according to the rules for function symbols commutes with $\beta\eta$-reduction. Thus $\to_R$ is confluent by Theorem 2.3.9 and the Hindley-Rosen Lemma 2.3.3. The remaining two conditions follow by straightforward transfinite induction, like in the proof of Lemma 4.2.11. □

Definition 5.2.13. Define $\mathcal{M}_\mathcal{A} = \langle \mathcal{C}, I, \mathcal{T}, \mathcal{F} \rangle$ where:

- $\mathcal{C}$ is the extensional first-order illative combinatory algebra constructed from the R-equality equivalence classes of terms, with $\mathsf{k} = [\mathsf{K}]$, $\mathsf{s} = [\mathsf{S}]$, $\mathsf{p} = [\mathsf{P}]$, etc., where by $[X]$ we denote the equivalence class of X,
- I is defined by $I(c) = [c]$ for $c \in \Sigma$,
- $\mathcal{T} = \{[X] \mid X \rightsquigarrow \top\}$,
- $\mathcal{F} = \{[X] \mid X \rightsquigarrow \bot\}$.

Theorem 5.2.14. *The structure $\mathcal{M}_\mathcal{A}$ is a classical $\mathcal{I}\mathrm{K}$-model such that for every $a \in \mathcal{A}$ there is $\overline{a} \in \mathcal{C}$, for every relation symbol r there is $\overline{r} \in \mathcal{C}$, and for every function symbol f there is $\overline{f} \in \mathcal{C}$, so that for $a_1, \ldots, a_n, a \in \mathcal{A}$:*

- $\overline{f} \cdot \overline{a_1} \cdot \ldots \cdot \overline{a_n} = \overline{a}$ *iff* $f^\mathcal{A}(a_1, \ldots, a_n) = a$,
- $\overline{r} \cdot \overline{a_1} \cdot \ldots \cdot \overline{a_n} \in \mathcal{T}$ *iff* $r^\mathcal{A}(a_1, \ldots, a_n)$ *holds,*
- $\overline{r} \cdot \overline{a_1} \cdot \ldots \cdot \overline{a_n} \in \mathcal{F}$ *iff* $r^\mathcal{A}(a_1, \ldots, a_n)$ *does not hold.*

Proof. Using Lemma 4.2.11 it is easy to check the conditions from Definition 5.1.12. The additional conditions in the statement of the theorem hold by construction. □

Corollary 5.2.15. *The system $\mathcal{I}\mathrm{K}$ is consistent, i.e., $\nvdash_{\mathcal{I}\mathrm{K}} \bot$.*

5.3 Translations

In this section we show sound and complete syntactic translations of traditional systems of first-order logic into corresponding illative systems. The proofs are similar to the ones in Section 4.3.

We adopt the notational conventions like in the previous section, i.e., X, Y, Z, etc., stand for terms in $\mathbb{T}$. Also t, s, etc., stand for first-order terms, φ, ψ, etc., stand for first-order formulas, and Δ, Δ', etc., stand for sets of first-order formulas. We assume that all function and relation symbols of traditional systems occur as constants in $\mathbb{T}$, and all variables of traditional systems occur as variables in $\mathbb{T}$. Sometimes we write, e.g., Δ, φ instead of $\Delta \cup \{\varphi\}$.

Recall the following abbreviations from Section 1.1.

$$\begin{aligned} \mathsf{F} &\equiv \lambda xyf.\Xi x(\lambda z.y(fz)) \\ \mathsf{F}_0 &\equiv \mathsf{I} \\ \mathsf{F}_{n+1} &\equiv \lambda x_1 \ldots x_{n+1}y.\mathsf{F}x_1(\mathsf{F}_n x_2 \ldots x_{n+1}y) \end{aligned}$$

Intuitively, $\mathsf{F}XYF$ means that F is a function from X to Y, and $\mathsf{F}_n X_1 \ldots X_n Y F$ means that F is an n-argument function from $X_1, \ldots, X_n$ to Y.

Definition 5.3.1. We define a mapping $\lceil - \rceil$ from first-order terms and formulas to the set of terms $\mathbb{T}$ of illative systems, and a context-providing mapping $\Gamma(-)$ from sets of first-order terms and formulas to sets of terms from $\mathbb{T}$. The definition of $\lceil - \rceil$ is by induction of the structure of its argument:

- $\lceil x \rceil \equiv x$ for x a variable,
- $\lceil f(t_1, \ldots, t_n) \rceil \equiv f \lceil t_1 \rceil \ldots \lceil t_n \rceil$,
- $\lceil r(t_1, \ldots, t_n) \rceil \equiv r \lceil t_1 \rceil \ldots \lceil t_n \rceil$,
- $\lceil \bot \rceil \equiv \bot$,
- $\lceil \varphi \vee \psi \rceil \equiv \lceil \varphi \rceil \vee \lceil \psi \rceil$,
- $\lceil \varphi \wedge \psi \rceil \equiv \lceil \varphi \rceil \wedge \lceil \psi \rceil$,
- $\lceil \varphi \to \psi \rceil \equiv \lceil \varphi \rceil \supset \lceil \psi \rceil$,
- $\lceil \forall x.\varphi \rceil \equiv \Xi \mathsf{A} \lambda x.\lceil \varphi \rceil$,
- $\lceil \exists x.\varphi \rceil \equiv \mathsf{X} \mathsf{A} \lambda x.\lceil \varphi \rceil$.

We extend the mapping $\lceil - \rceil$ to sets of first-order formulas thus: $\lceil \Delta \rceil = \{\lceil \varphi \rceil \mid \varphi \in \Delta\}$.

For a set of first-order terms and formulas Δ, the set $\Gamma(\Delta)$ is defined to contain:

- $\mathsf{F}_n \mathsf{A} \ldots \mathsf{A} \mathsf{H} r$ for each relation symbol r of arity n, where A occurs n times,
- $\mathsf{F}_n \mathsf{A} \ldots \mathsf{A} \mathsf{A} f$ for each function symbol f of arity n, where A occurs $n+1$ times,
- $\mathsf{A}x$ for each $x \in \mathrm{FV}(\Delta)$,
- $\mathsf{A}y$ for a fresh variable y, i.e., we assume y not to occur free in any first-order formula.[5]

[5]Note that the set of variables of $\mathbb{T}$ is distinct from the set of first-order variables. We assume each first-order variable to occur as a variable in $\mathbb{T}$, but not vice versa.

The last point is necessary, because in ordinary logic the universe is implicitly assumed to be non-empty. If we did not include the last point, we would not be able to prove soundness of the translations. In the proof of Theorem 5.3.4 we would not be able to transform an arbitrary Kripke $\mathcal{I}$J-model into a Kripke NJ-model, because some of the universes $\mathcal{A}_s$ might turn out to be empty. The proof of Theorem 5.3.6 would break down in the case for implication elimination.

Lemma 5.3.2. $\Gamma(\{t\}) \vdash_{\mathcal{I}\mathrm{J}} \mathsf{A}\lceil t \rceil$.

Proof. Induction on the structure of t. □

Lemma 5.3.3. $\Gamma(\{\varphi\}) \vdash_{\mathcal{I}\mathrm{J}} \mathsf{H}\lceil \varphi \rceil$.

Proof. Induction on the structure of φ, using Lemma 5.3.2. □

Theorem 5.3.4 (Completeness of the translation for $\mathcal{I}$J)**.**
$\Delta \Vdash_{\mathrm{NJ}} \varphi$ *iff* $\Gamma(\Delta, \varphi), \lceil \Delta \rceil \Vdash_{\mathcal{I}\mathrm{J}} \lceil \varphi \rceil$.

Proof. Assume $\Delta \Vdash_{\mathrm{NJ}} \varphi$. Let $\mathcal{M} = \langle \mathcal{C}, I, S, \leq, \sigma_0, \sigma_1 \rangle$ be a Kripke $\mathcal{I}$J-model, $s_0 \in S$ and ρ an $\mathcal{M}$-valuation such that $\mathcal{M}, s_0, \rho \Vdash \Gamma(\Delta, \varphi), \lceil \Delta \rceil$. We define a Kripke NJ-model

$$\mathcal{S} = \langle S, \leq, \{\mathcal{A}_s\} \rangle$$

by taking S and $\leq$ to be the same as in $\mathcal{M}$, and defining $\mathcal{A}_s = \langle A_s, \{f_i^{\mathcal{A}_s}\}, \{r_i^{\mathcal{A}_s}\} \rangle$ by:

- $A_s = \{c \in \mathcal{C} \mid s \in \sigma_1(\mathsf{A} \cdot c)\}$,
- $f^{\mathcal{A}_s}(a_1, \ldots, a_n) = I(f) \cdot a_1 \cdot \ldots \cdot a_n$,
- $r^{\mathcal{A}_s}(a_1, \ldots, a_n)$ holds iff $s \in \sigma_1(I(r) \cdot a_1 \cdot \ldots \cdot a_n)$.

Note that $\mathcal{A}_s \neq \emptyset$ for $s \in S$. This is because $\mathsf{A}y$ is present in $\Gamma(\Delta, \varphi)$ for a fresh variable y, so for each $s \geq s_0$ there exists $a \in \mathcal{C}$ such that $s \in \sigma_1(\mathsf{A} \cdot a)$. Hence $\mathcal{S}$ is a well-defined Kripke NJ-model.

For v an $\mathcal{M}$-valuation and $s \geq s_0$, we define an $\mathcal{A}_s$-valuation v_s by: $v_s(x) = v(x)$ for $x \in \mathrm{FV}(\Delta, \varphi)$, and $v_s(x) = a$ for other variables $x \notin \mathrm{FV}(\Delta, \varphi)$ and some arbitrary $a \in \mathcal{A}_s$. This is well-defined, because $\mathsf{A}x$ is present in $\Gamma(\Delta, \varphi)$ for $x \in \mathrm{FV}(\Delta, \varphi)$.

First, by induction on the structure of a term t such that $\mathrm{FV}(t) \subseteq \mathrm{FV}(\Delta, \varphi)$ we show for $s \geq s_0$:

(a) $[\![t]\!]_v^{\mathcal{M}} = [\![t]\!]_{v_s}^{\mathcal{A}_s}$.

Then, by induction on the structure of a subformula ψ of a formula from $\Delta \cup \{\varphi\}$, we prove that for $s \geq s_0$ we have:

(b) $\mathcal{S}, s, v_s \Vdash \psi$ iff $\mathcal{M}, s, v \Vdash \lceil \psi \rceil$.

For example, we show the case $\psi \equiv \forall x.\psi'$. Other cases are similar, with Lemma 5.3.3 and Theorem 5.1.7 needed for implication. We also need to use (a) for the base case when $\psi \equiv r(t_1, \ldots, t_n)$. Assuming $\psi \equiv \forall x.\psi'$ we have $\lceil \psi \rceil \equiv \Xi\mathsf{A}(\lambda x.\lceil \psi' \rceil)$.

Suppose $\mathcal{S}, s, v_s \Vdash \forall x.\psi'$. Let $s' \geq s$ and $a \in \mathcal{C}$ be such that $s' \in \sigma_1(\mathsf{A} \cdot a)$. Then $a \in \mathcal{A}_{s'}$, so $\mathcal{S}, s', v'_{s'} \Vdash \psi'$ where $v'(y) = v(y)$ for $x \neq y$ and $v'(x) = a$. By the inductive hypothesis $\mathcal{M}, s', v' \Vdash \lceil \psi' \rceil$. Since $s' \geq s$ was arbitrary, and of course $\mathcal{M}, s \Vdash \mathsf{LA}$, we obtain $\mathcal{M}, s, v \Vdash \Xi\mathsf{A}(\lambda x.\lceil \psi' \rceil)$.

Now assume $\mathcal{M}, s, v \Vdash \lceil \forall x.\psi \rceil$, i.e., $\mathcal{M}, s, v \Vdash \Xi\mathsf{A}(\lambda x.\lceil \psi' \rceil)$. Let $s' \geq s$ and $a \in \mathcal{A}_{s'}$. Then $\mathcal{M}, s', v' \Vdash \lceil \psi' \rceil$ where $v'(y) = v(y)$ for $y \neq x$ and $v'(x) = a$. By the inductive hypothesis $\mathcal{S}, s', v'_{s'} \Vdash \psi'$. This implies that $\mathcal{S}, s \Vdash \forall x.\psi'$.

Hence, by (b), we have $\mathcal{S}, s_0, \rho_s \Vdash \Delta$, because $\mathcal{M}, s_0, \rho \Vdash \lceil \Delta \rceil$. Thus $\mathcal{S}, s_0, \rho_s \Vdash \varphi$. This in turn implies $\mathcal{M}, s_0, \rho \Vdash \lceil \varphi \rceil$. Since $\mathcal{M}$, s_0 and ρ were arbitrary satisfying $\mathcal{M}, s_0, \rho \Vdash \Gamma(\Delta, \varphi), \lceil \Delta \rceil$, we have $\Gamma(\Delta, \varphi), \lceil \Delta \rceil \Vdash_{\mathcal{I}\mathrm{J}} \lceil \varphi \rceil$.

Assume $\Gamma(\Delta, \varphi), \lceil \Delta \rceil \Vdash_{\mathcal{I}\mathrm{J}} \lceil \varphi \rceil$. Let $\mathcal{S} = \langle \mathcal{S}, \leq, \{\mathcal{A}_s \mid s \in \mathcal{S}\} \rangle$ be a Kripke NJ-model, $s_0 \in \mathcal{S}$ and ρ be an $\mathcal{A}_{s_0}$-valuation such that $\mathcal{S}, s_0, \rho \Vdash \Delta$. We construct a Kripke $\mathcal{I}$J-model $\mathcal{M}$ using Theorem 5.2.9. This model has the same states and state ordering as $\mathcal{S}$. By induction we show that it satisfies the following for $s \geq s_0$ and v an $\mathcal{A}_s$-valuation:

- $\mathcal{M}, s, \overline{v} \Vdash \mathsf{A}\lceil t \rceil$,
- $\mathcal{M}, s, \overline{v} \Vdash \mathsf{H}\lceil \psi \rceil$,
- $[\![t]\!]^{\mathcal{M}}_{\overline{v}} = \overline{[\![t]\!]^{\mathcal{A}_s}_{v}}$,
- $\mathcal{M}, s, \overline{v} \Vdash \lceil \psi \rceil$ iff $\mathcal{S}, s, v \Vdash \psi$,

where $\mathrm{FV}(t, \psi) \subseteq \mathrm{FV}(\Delta, \varphi)$, and $\overline{v}$ is an $\mathcal{M}$-valuation such that $\overline{v}(x) \equiv \overline{v(x)}$, where $\overline{a}$ for $a \in \mathcal{A}_s$ is like in Theorem 5.2.9. The proof is straightforward and we omit it.

Hence, we have $\mathcal{M}, s_0, \overline{\rho} \Vdash \lceil \Delta \rceil$, because $\mathcal{S}, s_0, \rho \Vdash \Delta$. If follows from the definition of $\mathcal{M}$ that also $\mathcal{M}, s_0, \overline{\rho} \Vdash \Gamma(\Delta, \varphi)$. Thus $\mathcal{M}, s_0, \overline{\rho} \Vdash \lceil \varphi \rceil$. So $\mathcal{S}, s_0, \rho \Vdash \varphi$. Since $\mathcal{S}$, s_0 and ρ were arbitrary satisfying $\mathcal{S}, s_0, \rho \Vdash \Delta$, we obtain $\Delta \Vdash_{\mathrm{NJ}} \varphi$. □

Corollary 5.3.5. $\Delta \vdash_{\mathrm{NJ}} \varphi$ *iff* $\Gamma(\Delta, \varphi), \lceil \Delta \rceil \vdash_{\mathcal{I}\mathrm{J}} \lceil \varphi \rceil$.

Proof. Follows from Theorem 5.3.4, Theorem 5.1.7, Theorem 5.1.11 and Theorem 2.4.10. □

Theorem 5.3.6 (Soundness of the translation for $\mathcal{I}$K)**.**
If $\Delta \vdash_{\mathrm{NK}} \varphi$ *then* $\Gamma(\Delta, \varphi), \lceil \Delta \rceil \vdash_{\mathcal{I}\mathrm{K}} \lceil \varphi \rceil$.

Proof. Because we have not proved completeness of classical $\mathcal{I}$K-models for $\mathcal{I}$K, the proof needs to be done syntactically, using Lemma 5.3.3. We proceed by induction on the length of derivation of $\Delta \vdash_{\mathrm{NK}} \varphi$. The interesting case is with implication elimination. So assume $\Delta \vdash_{\mathrm{NK}} \varphi$ was obtained from $\Delta \vdash_{\mathrm{NK}} \psi$ and $\Delta \vdash_{\mathrm{NK}} \psi \to \varphi$. By the inductive hypothesis

$$\Gamma(\Delta, \varphi, \psi), \lceil \Delta \rceil \vdash_{\mathcal{I}\mathrm{K}} \lceil \psi \rceil \supset \lceil \varphi \rceil$$

and also $\Gamma(\Delta, \psi), \lceil \Delta \rceil \vdash_{\mathcal{I}\mathrm{K}} \lceil \psi \rceil$. Since $\Gamma(\Delta, \psi) \subseteq \Gamma(\Delta, \varphi, \psi) = \Gamma(\Delta, \varphi) \cup \Gamma(\psi)$ we obtain

$$\Gamma(\Delta, \varphi), \Gamma(\psi), \lceil \Delta \rceil \vdash_{\mathcal{I}\mathrm{K}} \lceil \varphi \rceil.$$

We have $\Gamma(\psi) \setminus \Gamma(\Delta, \varphi) = \mathsf{A}x_1, \ldots, \mathsf{A}x_n$ for some $x_1, \ldots, x_n \in \mathrm{FV}(\psi) \setminus \mathrm{FV}(\Delta, \varphi)$. Recall that $\mathsf{A}y \in \Gamma(\Delta, \psi)$ for a fresh variable $y \notin \mathrm{FV}(\Delta, \varphi, \psi)$. Substituting y for each x_i, by (Sub) from Lemma 4.1.2 we have $\Gamma(\Delta, \varphi), \mathsf{A}y, \ldots, \mathsf{A}y, \lceil \Delta \rceil \vdash_{\mathcal{I}\mathrm{K}} \lceil \varphi \rceil$, i.e., $\Gamma(\Delta, \varphi) \vdash_{\mathcal{I}\mathrm{K}} \lceil \varphi \rceil$. □

Theorem 5.3.7 (Completeness of the translation for $\mathcal{I}$K)**.**
If $\Gamma(\Delta, \varphi), \lceil\Delta\rceil \models_{\mathcal{I}\mathrm{K}} \lceil\varphi\rceil$ *then* $\Delta \models_{\mathrm{NK}} \varphi$.

Proof. Assume $\Gamma(\Delta, \varphi), \lceil\Delta\rceil \models_{\mathcal{I}\mathrm{K}} \lceil\varphi\rceil$. Let $\mathcal{A}$ be a classical NK-structure and ρ an $\mathcal{A}$-valuation such that $\mathcal{A}, v \models \Delta$. Take $\mathcal{M}$ to be the classical $\mathcal{I}$K-model obtained by applying Theorem 5.2.14 to $\mathcal{A}$. It is easy to check by induction on the structure of a formula ψ that:

- $\mathcal{M}, \overline{v} \models \mathsf{A}\lceil t\rceil$,
- $\mathcal{M}, \overline{v} \models \mathsf{H}\lceil\psi\rceil$,
- $[\![t]\!]^{\mathcal{M}}_{\overline{v}} = \overline{[\![t]\!]^{\mathcal{A}}_{v}}$,
- $\mathcal{M}, \overline{v} \models \lceil\psi\rceil$ iff $\mathcal{A}, v \models \psi$,

where $\overline{v}$ is an $\mathcal{M}$-valuation such that $\overline{v}(x) \equiv \overline{v(x)}$, where $\overline{a}$ for $a \in \mathcal{A}$ is as in Theorem 5.2.14. Then we have $\mathcal{M}, \overline{\rho} \models \Gamma(\Delta, \varphi), \lceil\Delta\rceil$. Thus $\mathcal{M}, \overline{\rho} \models \lceil\varphi\rceil$. Hence $\mathcal{A}, \rho \models \varphi$. Therefore, $\Delta \models_{\mathrm{NK}} \varphi$. □

Corollary 5.3.8. *If* $\Gamma(\Delta, \varphi), \lceil\Delta\rceil \vdash_{\mathcal{I}\mathrm{K}} \lceil\varphi\rceil$ *then* $\Delta \vdash_{\mathrm{NK}} \varphi$.

Proof. Follows from Theorem 5.3.7, Theorem 5.1.13 and Theorem 2.4.8. □

Chapter 6

Higher-order predicate logic

6.1 Illative systems

Definition 6.1.1. The system $\mathcal{I}\mathrm{K}\omega$ of *intensional classical higher-order illative combinatory logic* comes in three variants $\mathcal{I}\mathrm{K}\omega_{\lambda\beta\eta}$, $\mathcal{I}\mathrm{K}\omega_{\lambda\beta}$ and $\mathcal{I}\mathrm{K}\omega_{\mathrm{CL}w}$ which differ in the underlying reduction system. As in the preceeding chapters, we shall only give definitions and proofs for $\mathcal{I}\mathrm{K}\omega_{\lambda\beta\eta}$, and possibly note the differences with other variants. The set of terms $\mathbb{T}$ is defined separately for each variant, basing on a signature Σ containing the following illative constants: Ξ, $\mathsf{\Lambda}$, V, $\neg$, $\bot$, and a constant A_τ for each $\tau \in \mathcal{B}$, where $\mathcal{B}$ is some specific set of *base types*. We adopt the abbreviations (see also Section 1.1):

- $\top \equiv \neg\bot$,
- $X \wedge Y \equiv \mathsf{\Lambda} XY$,
- $X \vee Y \equiv \mathsf{V} XY$,
- $X \supset Y \equiv \neg X \vee Y$,
- $\mathsf{H} \equiv \lambda x.x \vee \neg x$,
- $\mathsf{L} \equiv \lambda x.\Xi xx$,
- $\mathsf{X} \equiv \lambda xy.\neg(\Xi x(\lambda z.\neg(yz)))$,
- $\forall x : X \,.\, Y \equiv \Xi X(\lambda x.Y)$ where $x \notin \mathrm{FV}(X)$,
- $\exists x : X \,.\, Y \equiv \mathsf{X} X(\lambda x.Y)$ where $x \notin \mathrm{FV}(X)$,
- $\mathsf{F} \equiv \lambda xyf.\Xi x(\lambda z.y(fz))$,
- $\mathsf{F}_0 \equiv \mathsf{I}$,
- $\mathsf{F}_{n+1} \equiv \lambda x_1 \ldots x_{n+1} y.\mathsf{F} x_1(\mathsf{F}_n x_2 \ldots x_{n+1} y)$,
- $A \to B \equiv \mathsf{F} AB$,
- $\mathsf{Q}_L \equiv \lambda axy.\forall p : a \to \mathsf{H} \,.\, px \supset py$,
- $X =_A Y \equiv \mathsf{Q}_L AXY$

The rules of $\mathcal{I}\mathrm{K}\omega$ are those of Figure 4.3 and Figure 6.1.

The illative primitive Q_L represents typed Leibniz equality: $\mathsf{Q}_L A$ denotes Leibniz equality in type A. As mentioned above, we usually write $X =_A Y$ instead of $\mathsf{Q}_L AXY$. The system $\mathcal{I}\mathrm{K}\omega$ may be extended to a system $e\mathcal{I}\mathrm{K}\omega$ extensional wrt. Q_L by adding the following rules:

$$\frac{\Gamma \vdash \forall x : A \,.\, Xx =_B Yx \quad x \notin \mathrm{FV}(X, Y, A, B)}{\Gamma \vdash X =_{A \to B} Y} \ (\mathrm{Ext}_f)$$

$$\frac{\Gamma \vdash X \supset Y \quad \Gamma \vdash Y \supset X}{\Gamma \vdash X =_H Y} \ (\mathrm{Ext}_b)$$

$$\frac{\Gamma, Xx \vdash Yx \quad \Gamma \vdash \mathsf{L}X \quad x \notin \mathrm{FV}(\Gamma, X, Y)}{\Gamma \vdash \Xi XY} \ (\Xi\mathrm{I}) \qquad \frac{\Gamma \vdash \Xi XY \quad \Gamma \vdash XZ}{\Gamma \vdash YZ} \ (\Xi\mathrm{E})$$

$$\frac{\Gamma \vdash XZ \quad \Gamma \vdash \neg(YZ) \quad \Gamma \vdash \mathsf{L}X}{\Gamma \vdash \neg(\Xi XY)} \ (\neg\Xi\mathrm{I})$$

$$\frac{\Gamma \vdash \neg(\Xi XY) \quad \Gamma, Xx, \neg(Yx) \vdash Z \quad x \notin \mathrm{FV}(\Gamma, X, Y, Z)}{\Gamma \vdash Z} \ (\neg\Xi\mathrm{E})$$

$$\frac{\Gamma, Xx \vdash \mathsf{H}(Yx) \quad \Gamma \vdash \mathsf{L}X \quad x \notin \mathrm{FV}(\Gamma, X, Y)}{\Gamma \vdash \mathsf{H}(\Xi XY)} \ (\Xi\mathsf{HI})$$

$$\frac{\Gamma \vdash \mathsf{H}(\Xi XY)}{\Gamma \vdash \mathsf{L}X} \ (\Xi\mathsf{LE})$$

$$\frac{\tau \in \mathcal{B}}{\Gamma \vdash \mathsf{L}\mathsf{A}_\tau} \ (\mathsf{A}_\tau\mathsf{L}) \qquad \frac{}{\Gamma \vdash \mathsf{L}\mathsf{H}} \ (\mathsf{HL}) \qquad \frac{\Gamma \vdash \mathsf{L}X \quad \Gamma, Xx \vdash \mathsf{L}Y \quad x \notin \mathrm{FV}(\Gamma, X, Y)}{\Gamma \vdash \mathsf{L}(\mathsf{F}XY)} \ (\mathsf{FL})$$

Figure 6.1: Additional rules of $\mathcal{I}\mathrm{K}\omega$

Lemma 6.1.2. *The rules from Figure 4.1, rule* (EM) *from Definition 4.1.1, and rules* (XI), (XE), (XHI) *and* (XLE) *form Figure 5.1, are all admissible in* $\mathcal{I}\mathrm{K}\omega$.

Proof. Straightforward, using Theorem 4.1.18. □

Lemma 6.1.3. *If* $\Gamma \vdash X =_A Y$, $\Gamma \vdash \mathsf{F}ABZ$, $\Gamma \vdash AX$ *and* $\Gamma \vdash \mathsf{L}B$, *then* $\Gamma \vdash ZX =_B ZY$.

Proof. Assume $\Gamma \vdash X =_A Y$, $\Gamma \vdash \mathsf{F}ABZ$, $\Gamma \vdash \mathsf{L}A$ and $\Gamma \vdash \mathsf{L}B$. Since $\Gamma \vdash \mathsf{L}B$, by $(\Xi\mathrm{I})$ it suffices to show

$$(\star) \qquad \Gamma, \mathsf{F}B\mathsf{H}p \vdash p(ZX) \supset p(ZY).$$

Let $M \equiv \lambda x.p(Zx)$. We have $\Gamma, \mathsf{F}B\mathsf{H}p \vdash \mathsf{F}A\mathsf{H}M$, because $\Gamma \vdash \mathsf{F}ABZ$. Hence

$$\Gamma, \mathsf{F}B\mathsf{H}p \vdash MX \supset MY$$

because $\Gamma \vdash X =_A Y$. Therefore

$$\Gamma, \mathsf{F}B\mathsf{H}p, p(ZX) \vdash p(ZY)$$

by (PE) and (Eq). Because $\Gamma \vdash AX$ and $\Gamma \vdash \mathsf{F}ABZ$, we have $\Gamma, \mathsf{F}B\mathsf{H}p \vdash \mathsf{H}(p(ZX))$. Hence we obtain ($\star$) by ($\mathsf{PI}$). $\square$

Definition 6.1.4. A *higher-order illative combinatory algebra* (HOICA) is a tuple

$$\langle \mathcal{C}, \cdot, \mathsf{k}, \mathsf{s}, \wedge, \vee, \neg, \bot, \Xi, \{\mathsf{A}_\tau\}_{\tau \in \mathcal{B}} \rangle$$

where $\langle \mathcal{C}, \cdot, \mathsf{k}, \mathsf{s} \rangle$ is an extensional combinatory algebra, $\wedge, \vee, \neg, \bot, \Xi \in \mathcal{C}$ and $\mathsf{A}_\tau \in \mathcal{C}$ for $\tau \in \mathcal{B}$. In other words, a higher-order illative combinatory algebra is an extensional combinatory algebra with additional distinguished elements. We often confuse a HOICA with its carrier set $\mathcal{C}$. In a HOICA $\mathcal{C}$ we define the elements h, p, etc., by the following equations, for arbitrary $a, b, c \in \mathcal{C}$:

- $\mathsf{h} \cdot a = \vee \cdot (\neg \cdot a) \cdot a$,
- $\mathsf{p} \cdot a \cdot b = \vee \cdot (\neg \cdot a) \cdot b$,
- $\mathsf{L} \cdot a = \Xi \cdot a \cdot a$,
- $\mathsf{x} \cdot a \cdot b = \neg \cdot (\Xi \cdot a \cdot (\mathsf{s} \cdot (\mathsf{k} \cdot \neg) \cdot b))$,
- $\mathsf{f} \cdot a \cdot b \cdot c = \Xi \cdot a \cdot (\mathsf{s} \cdot (\mathsf{k} \cdot b) \cdot c)$,
- $\mathsf{q} \cdot a \cdot b \cdot c = \Xi \cdot (\mathsf{f} \cdot a \cdot \mathsf{h}) \cdot e$,

where $e \in \mathcal{C}$ is the unique element such that

$$e \cdot d = \mathsf{p} \cdot (d \cdot b) \cdot (d \cdot c)$$

for any $d \in \mathcal{C}$. Note that the above equations uniquely define elements of $\mathcal{C}$, because $\mathcal{C}$ is extensional.

Definition 6.1.5. An $\mathcal{I}\mathrm{K}\omega$*-model* is a tuple $\langle \mathcal{C}, I, \mathcal{T}, \mathcal{F} \rangle$ where:

- $\mathcal{C}$ is a higher-order illative combinatory algebra.
- I is a function from Σ to $\mathcal{C}$.
- $\mathcal{T}$ and $\mathcal{F}$ are sets of elements of $\mathcal{C}$ satisfying the following for any $a, b \in \mathcal{C}$, where we use the notation $\mathcal{T}(a) = \{b \mid a \cdot b \in \mathcal{T}\}$ for $a \in \mathcal{C}$.
    1. $\mathcal{T} \cap \mathcal{F} = \emptyset$,
    2. $\bot \in \mathcal{F}$,
    3. $\neg \cdot a \in \mathcal{T}$ iff $a \in \mathcal{F}$,

4. $\neg \cdot a \in \mathcal{F}$ iff $a \in \mathcal{T}$,
5. $\mathsf{v} \cdot a \cdot b \in \mathcal{T}$ iff $a \in \mathcal{T}$ or $b \in \mathcal{T}$,
6. $\mathsf{v} \cdot a \cdot b \in \mathcal{F}$ iff $a \in \mathcal{F}$ and $b \in \mathcal{F}$,
7. $\wedge \cdot a \cdot b \in \mathcal{T}$ iff $a \in \mathcal{T}$ and $b \in \mathcal{T}$,
8. $\wedge \cdot a \cdot b \in \mathcal{F}$ iff $a \in \mathcal{F}$ or $b \in \mathcal{F}$,
9. $\Xi \cdot a \cdot b \in \mathcal{T}$ iff $\mathsf{L} \cdot a \in \mathcal{T}$ and for every $c \in \mathcal{C}$ with $a \cdot c \in \mathcal{C}$ we have $b \cdot c \in \mathcal{C}$,
10. $\Xi \cdot a \cdot b \in \mathcal{F}$ iff $\mathsf{L} \cdot a \in \mathcal{T}$ and there exists $c \in \mathcal{C}$ with $a \cdot c \in \mathcal{T}$ and $b \cdot c \in \mathcal{F}$,
11. $\mathsf{L} \cdot \mathsf{h} \in \mathcal{T}$,
12. $\mathsf{L} \cdot \mathsf{A}_\tau \in \mathcal{T}$ for each $\tau \in \mathcal{B}$,
13. if $\mathsf{L} \cdot a \in \mathcal{T}$, and $\mathcal{T}(a) = \emptyset$ or $\mathsf{L} \cdot b \in \mathcal{T}$, then $\mathsf{L} \cdot (\mathsf{f} \cdot a \cdot b) \in \mathcal{T}$.

An $e\mathcal{I}\mathrm{K}\omega$-*model* is an $\mathcal{I}\mathrm{K}\omega$-model additionally satisfying the following for all $a, b, c, d \in \mathcal{C}$:

14. if $\mathsf{L} \cdot a \in \mathcal{T}$ and for every $e \in \mathcal{T}(a)$ we have $\mathsf{q} \cdot b \cdot (c \cdot e) \cdot (d \cdot e) \in \mathcal{T}$, then $\mathsf{q} \cdot (\mathsf{f} \cdot a \cdot b) \cdot c \cdot d \in \mathcal{T}$,
15. if $a, b \in \mathcal{T}$ or $a, b \in \mathcal{F}$ then $\mathsf{q} \cdot \mathsf{h} \cdot a \cdot b \in \mathcal{T}$.

Let $\mathcal{M}$ be an $\mathcal{I}\mathrm{K}\omega$-model or an $e\mathcal{I}\mathrm{K}\omega$-model. An $\mathcal{M}$-*valuation* is a function from V to $\mathcal{C}$ (cf. Definition 2.3.17). Given an $\mathcal{M}$-valuation $\rho : V \to \mathcal{C}$ we define the *value* of $M \in \mathbb{T}_{\mathrm{CL}}$, denoted $[\![M]\!]^{\mathcal{M}}_\rho$ or just $[\![M]\!]_\rho$, by induction on the structure of M:

- $[\![x]\!]_\rho = \rho(x)$ if $x \in V$,
- $[\![\mathsf{K}]\!]_\rho = \mathsf{k}$, $[\![\mathsf{S}]\!]_\rho = \mathsf{s}$,
- $[\![\neg]\!]_\rho = \neg$, $[\![\mathsf{V}]\!]_\rho = \mathsf{v}$, $[\![\mathsf{\Lambda}]\!]_\rho = \wedge$, $[\![\bot]\!]_\rho = \bot$, $[\![\Xi]\!]_\rho = \Xi$,
- $[\![c]\!]_\rho = I(c)$ if $c \in \Sigma \setminus \{\neg, \mathsf{V}, \mathsf{\Lambda}, \bot, \Xi\}$,
- $[\![M_1 M_2]\!]_\rho = [\![M_1]\!]_\rho \cdot [\![M_1]\!]_\rho$.

For $M \in \mathbb{T}_\lambda$ we set $[\![M]\!]_\rho = [\![(M)_{\mathrm{CL}}]\!]_\rho$.

If $[\![M]\!]^{\mathcal{M}}_\rho \in \mathcal{T}$, we write $\mathcal{M}, \rho \models M$. If M is closed then we write $\mathcal{M} \models M$. We write $\mathcal{M}, \rho \models \Gamma$ if $\mathcal{M}, \rho \models M$ for all $M \in \Gamma$. Finally, we write $\Gamma \models_{\mathcal{I}\mathrm{K}\omega} M$ (resp. $\Gamma \models_{e\mathcal{I}\mathrm{K}\omega} M$) if for every $\mathcal{I}\mathrm{K}\omega$-model (resp. $e\mathcal{I}\mathrm{K}\omega$-model) $\mathcal{M}$ and every $\mathcal{M}$-valuation ρ, the condition $\mathcal{M}, \rho \models \Gamma$ implies $\mathcal{M}, \rho \models M$.

Lemma 6.1.6. *If* $\rho' = \rho[x/[\![X]\!]_\rho]$ *then* $[\![Y]\!]_{\rho'} = [\![Y[x/X]]\!]_\rho$.

Proof. Analogous to Lemma 5.1.5. ☐

Theorem 6.1.7. *If* $\Gamma \vdash_{\mathcal{I}} X$ *then* $\Gamma \models_{\mathcal{I}} X$, *where* $\mathcal{I} = \mathcal{I}\mathrm{K}\omega$ *or* $\mathcal{I} = e\mathcal{I}\mathrm{K}\omega$.

Proof. Straightforward induction on the length of derivation of $\Gamma \vdash_{\mathcal{I}} X$. ☐

6.2 Model construction

In this section we construct a model for $\mathcal{I}\mathrm{K}\omega$ and $e\mathcal{I}\mathrm{K}\omega$, thus establishing consistency of these systems. The model construction is parameterised by a standard model

$$\mathcal{N} = \langle\{\mathcal{D}_\tau \mid \tau \in \mathscr{T}\}, I\rangle$$

for higher-order logic (see Definition 2.4.13). We assume the set of base types $\mathcal{B}$ in the model $\mathcal{N}$ to be the same as the set of base types of $\mathcal{I}\mathrm{K}\omega$, and that all constants of $\mathrm{NK}\omega$ are present in the syntax of $\mathcal{I}\mathrm{K}\omega$. For the model construction, we also assume that each element $d \in \mathcal{D}_\tau$ for any $\tau \in \mathscr{T}$ occurs as a distinct constant in the set of terms $\mathbb{T}$. If $I(c) = d \in \mathcal{D}_\tau$ then without loss of generality we assume that $c \equiv d$. If $f \in \mathcal{D}_{\tau\to\rho}$ and $a \in \mathcal{D}_\tau$, then to avoid confusion with the term fa we write $f^{\mathcal{N}}(a)$ instead of $f(a)$ to denote the value of the function f at argument a. Without loss of generality, we identify the term $\bot$ (resp. $\top$) with the element $\bot$ (resp. $\top$) of $\mathcal{D}_o$. In this section we use the abbreviation $\mathsf{L}X \equiv \Xi XX$, i.e., $\mathsf{L}X$ stands for ΞXX and not for $(\lambda x.\Xi xx)X$. This convention is to shorten notations.

Definition 6.2.1. For $\tau \in \mathscr{T}$ and an ordinal α we define the representation relations $\succ^\alpha_\tau \in \mathbb{T}\times\mathbb{T}$, the contraction relation $\to^\alpha \in \mathbb{T}\times\mathbb{T}$, and the relation $\succ^\alpha_{\mathscr{T}} \in \mathbb{T}\times\mathscr{T}$ inductively. The notation $X \rightsquigarrow^\alpha_\tau Y$ stands for $X \stackrel{*}{\to}^\alpha \cdot \succ^\alpha_\tau Y$, and the notations $\succ^{<\alpha}_\tau$, $\rightsquigarrow^{<\alpha}_\tau$ are defined as usual.

- (β) $(\lambda x.X)Y \to^\alpha X[x/Y]$,
- (η) $\lambda x.Xx \to^\alpha X$ if $x \notin \mathrm{FV}(X)$,
- (γ) $fX \to^\alpha b$ if $f \in \mathcal{D}_{\tau_1\to\tau_2}$, $b \in \mathcal{D}_{\tau_2}$, $f^{\mathcal{N}}(a) = b$ and $X \succ^{<\alpha}_{\tau_1} a$, for some $a \in \mathcal{D}_{\tau_1}$,
- $(\mathcal{D}_\tau)$ $d \succ^\alpha_\tau d$ for $d \in \mathcal{D}_\tau$ and $\tau \in \mathcal{B}\cup\{o\}$,
- $(\mathcal{F}_\tau)$ $X \succ^\alpha_\tau d$ if $\tau = \tau_1 \to \tau_2$, $d \in \mathcal{D}_{\tau_1\to\tau_2}$ and for every $a \in \mathcal{D}_{\tau_1}$ we have $Xa \rightsquigarrow^{<\alpha}_{\tau_2} d^{\mathcal{N}}(a)$,
- $(\neg_\top)$ $\neg X \succ^\alpha_o \top$ if $X \succ^{<\alpha}_o \bot$,
- $(\neg_\bot)$ $\neg X \succ^\alpha_o \bot$ if $X \succ^{<\alpha}_o \top$,
- $(\mathsf{V}_\top)$ $X \vee Y \succ^\alpha_o \top$ if $X \succ^{<\alpha}_o \top$ or $Y \succ^{<\alpha}_o \top$,
- $(\mathsf{V}_\bot)$ $X \vee Y \succ^\alpha_o \bot$ if $X \succ^{<\alpha}_o \bot$ and $Y \succ^{<\alpha}_o \bot$,
- $(\mathsf{\Lambda}_\top)$ $X \wedge Y \succ^\alpha_o \top$ if $X \succ^{<\alpha}_o \top$ and $Y \succ^{<\alpha}_o \top$,
- $(\mathsf{\Lambda}_\bot)$ $X \wedge Y \succ^\alpha_o \bot$ if $X \succ^{<\alpha}_o \bot$ or $Y \succ^{<\alpha}_o \bot$,
- $(\Xi_\top)$ $\Xi XY \succ^\alpha_o \top$ if $X \succ^{<\alpha}_{\mathscr{T}} \tau$ and for every $d \in \mathcal{D}_\tau$ we have $Yd \rightsquigarrow^{<\alpha}_o \top$,
- $(\Xi_\bot)$ $\Xi XY \succ^\alpha_o \bot$ if $X \succ^{<\alpha}_{\mathscr{T}} \tau$ and there exists $d \in \mathcal{D}_\tau$ with $Yd \rightsquigarrow^{<\alpha}_o \bot$,
- $(\mathsf{L}_\top)$ $\mathsf{L}X \succ^\alpha_o \top$ if $X \succ^{<\alpha}_{\mathscr{T}} \tau$ for some $\tau \in \mathscr{T}$,
- $(\mathsf{A}_\top)$ $\mathsf{A}_\tau d \succ^\alpha_o \top$ if $\tau \in \mathcal{B}$ and $d \in \mathcal{D}_\tau$,
- $(\mathsf{H}_{\mathscr{T}})$ $\mathsf{H} \succ^\alpha_{\mathscr{T}} o$,
- $(\mathsf{A}_{\mathscr{T}})$ $\mathsf{A}_\tau \succ^\alpha_{\mathscr{T}} \tau$ for $\tau \in \mathcal{B}$,

($\mathsf{F}_{\mathscr{T}}$) $\lambda f.\Xi X(\lambda x.Y[y/(fx)]) \succ^{\alpha}_{\mathscr{T}} \tau_1 \to \tau_2$ if $f, x \notin \mathrm{FV}(X, Y)$, $X \succ^{<\alpha}_{\mathscr{T}} \tau_1$ and $\lambda y.Y \rightsquigarrow^{<\alpha}_{\mathscr{T}} \tau_2$.

It is to be understood that the relation $\to^{\alpha}$ is the compatible closure of the rules (β), (η) and (γ), while the relations $\succ^{\alpha}_{\tau}$ for $\tau \in \mathscr{T}$ and $\succ^{\alpha}_{\mathscr{T}}$ are defined directly by the corresponding rules, i.e., without taking compatible closure – these are not contraction relations.

It is easy to see that for $\alpha \leq \kappa$ we have $\to^{\alpha} \subseteq \to^{\kappa}$, $\succ^{\alpha}_{\tau} \subseteq \succ^{\kappa}_{\tau}$ for $\tau \in \mathscr{T}$, and $\succ^{\alpha}_{\mathscr{T}} \subseteq \succ^{\kappa}_{\mathscr{T}}$. Hence by Theorem 2.1.3 there is the closure ordinal ζ with $\to^{\zeta} = \to^{<\zeta}$, $\succ^{\zeta}_{\tau} = \succ^{<\zeta}_{\tau}$ for $\tau \in \mathscr{T}$, and $\succ^{\zeta}_{\mathscr{T}} = \succ^{<\zeta}_{\mathscr{T}}$. We use the notations $\to$, $\succ_{\tau}$ ($\tau \in \mathscr{T}$), $\succ_{\mathscr{T}}$ for $\to^{\zeta}$, $\succ^{\zeta}_{\tau}$ ($\tau \in \mathscr{T}$), $\succ^{\zeta}_{\mathscr{T}}$, respectively.

By $\to^{\alpha}_{\gamma}$ we denote $\to^{\alpha} \setminus \to_{\beta\eta}$, and by $\to_{\gamma}$ we denote $\to \setminus \to_{\beta\eta}$. We sometimes write $\to_{\beta\eta\gamma}$ instead of $\to$ to avoid confusion with other reduction relations. The relation $\to_{\gamma}$ is called γ-contraction, and its transitive-reflexive closure $\xrightarrow{*}_{\gamma}$ is called γ-reduction.

We define the reduction system R by $R = \langle \to_{\beta\eta\gamma}, \{\succ_{\tau}\}_{\tau \in \mathscr{T}} \cup \{\succ_{\mathscr{T}}\}\rangle$. The reduction system R^{α} is defined by $R^{\alpha} = \langle \to^{\alpha}, \{\succ^{\alpha}_{\tau}\}_{\tau \in \mathscr{T}} \cup \{\succ^{\alpha}_{\mathscr{T}}\}\rangle$.

The intuition behind $\succ_{\tau}$ for $\tau \in \mathscr{T}$ is that $X \succ_{\tau} d$ means "X is represented by d in type τ", i.e., "X behaves exactly like d in every context where a value of type τ is expected". The closure under arbitrary contexts where a value of type τ is "expected" is essentially implemented by γ-reduction. The relation $X \succ_{\mathscr{T}} \tau$ is interpreted as "X intepreted as a type is represented by τ".

The rules for $\succ_{o}$ correspond to the conditions on $\mathcal{T}$ and $\mathcal{F}$ in Definition 6.1.5. They are as one would expect them to be, except perhaps the rules ($\Xi_{\top}$) and ($\Xi_{\bot}$). Instead of the rule ($\Xi_{\top}$) one might expect

($\Xi'_{\top}$) $\Xi XY \succ^{\alpha}_{o} \top$ if $\mathsf{L}X \succ^{<\alpha}_{o} \top$ and for all Z such that $XZ \rightsquigarrow^{<\alpha}_{o} \top$ we have $YZ \rightsquigarrow^{<\alpha}_{o} \top$.

However, in this rule there is a negative reference to $\rightsquigarrow^{<\alpha}_{o}$ in $XZ \rightsquigarrow^{<\alpha}_{o} \top$, so it may no longer be the case that $\succ^{\alpha}_{o} \subseteq \succ^{\kappa}_{o}$ for $\alpha \leq \kappa$, and we could not apply Theorem 2.1.3. The way we solve this major problem is to restrict quantification to constants from appropriate $\mathcal{D}_{\tau}$. We will show that if $X \succ_{\mathscr{T}} \tau$ then quantifying over only elements of $\mathcal{D}_{\tau}$ is equivalent to quantifying over all Z such that $XZ \rightsquigarrow_{o} \top$. A crucial step is to show that the reduction system R is invariant (see Section 2.3.3).

To see how the argument goes and where invariance is used, assume that $X \succ_{\mathscr{T}} \tau$ and for every $d \in \mathcal{D}_{\tau}$ we have $Yd \rightsquigarrow_{o} \top$. Suppose $XZ \rightsquigarrow_{o} \top$. We will show in one of the following lemmas that if $X \succ_{\mathscr{T}} \tau$, then $XZ \rightsquigarrow_{o} \top$ implies that there is $d \in \mathcal{D}_{\tau}$ with $Z \rightsquigarrow_{\tau} d$, i.e., that there is an element of $\mathcal{D}_{\tau}$ by which Z is represented in type τ. But since $Yd \rightsquigarrow_{o} \top$ and $Z \rightsquigarrow_{\tau} d$, by invariance (see Lemma 2.3.15) we then obtain $YZ \rightsquigarrow_{\tau} \top$. In other words, if Yd holds for all $d \in \mathcal{D}_{\tau}$, then also YZ holds for all terms of type τ (the terms of type τ are those Z such that $XZ \rightsquigarrow_{o} \top$, because X interpreted as a type is represented by τ).

The real problem here, and the reason we need an argument like the one sketched above, is with function types. For a base type $\tau \in \mathcal{B}$, the only terms which have type τ are the elements of $\mathcal{D}_{\tau}$ (more precisely, the constants in $\mathbb{T}$ corresponding to these elements). But the terms having a function type $\tau_1 \to \tau_2$ are defined "semantically": these are all terms X such that for any Y of type τ_1 the term XY has type τ_2.

We now give several examples to illustrate Definition 6.2.1.

Example 6.2.2. Let $\tau \in \mathcal{B}$ and let $\mathrm{id} \in \mathcal{D}_{\tau\to\tau}$ be the identity function on $\mathcal{D}_\tau$, i.e. $\mathrm{id}^{\mathcal{N}}(d) = d$ for $d \in \mathcal{D}_\tau$. We show $\lambda x.x \succ_{\tau\to\tau} \mathrm{id}$. For $d \in \mathcal{D}_\tau$ we have $(\lambda x.x)d \to_\beta d \succ_\tau d$, so $(\lambda x.x)d \rightsquigarrow_\tau d = \mathrm{id}^{\mathcal{N}}(d)$. Thus $\lambda x.x \succ_{\tau\to\tau} d$ by $(\mathcal{F}_{\tau\to\tau})$.

Let $f \in \mathcal{D}_{\tau\to\tau\to\tau}$ be such that $f^{\mathcal{N}}(d) = \mathrm{id}$ for $d \in \mathcal{D}_\tau$. We show $\lambda yx.x \succ_{\tau\to\tau\to\tau} f$. For $d \in \mathcal{D}_\tau$ we have $(\lambda yx.x)d \to_\beta \lambda x.x \succ_{\tau\to\tau} \mathrm{id}$, so $(\lambda yx.x)d \rightsquigarrow_{\tau\to\tau} \mathrm{id}$. Thus $\lambda yx.x \succ_{\tau\to\tau\to\tau} f$ by $(\mathcal{F}_{\tau\to\tau\to\tau})$.

Let $g \in \mathcal{D}_{((\tau\to\tau\to\tau)\to\tau)\to\tau}$ be such that $g^{\mathcal{N}}(d) = d^{\mathcal{N}}(f)$ for $d \in \mathcal{D}_{(\tau\to\tau\to\tau)\to\tau}$, where f is as in the previous paragraph. We show $\lambda z.z(\lambda yx.x) \succ_{((\tau\to\tau\to\tau)\to\tau)\to\tau} g$. For $d \in \mathcal{D}_{(\tau\to\tau\to\tau)\to\tau}$ we have $(\lambda z.z(\lambda yx.x))d \to_\beta d(\lambda yx.x) \to_\gamma d^{\mathcal{N}}(f)$ because $\lambda yx.x \succ_{\tau\to\tau\to\tau} f$. We also have $d^{\mathcal{N}}(f) \succ_\tau d^{\mathcal{N}}(f)$ by $(\mathcal{D}_\tau)$, so $(\lambda z.z(\lambda yx.x))d \rightsquigarrow_\tau d^{\mathcal{N}}(f) = g^{\mathcal{N}}(d)$. Thus we conclude $\lambda z.z(\lambda yx.x) \succ_{((\tau\to\tau\to\tau)\to\tau)\to\tau} g$ by $(\mathcal{F}_{((\tau\to\tau\to\tau)\to\tau)\to\tau})$.

Now we proceed with the model construction. We shall show that the reduction system R is closed under substitution (see Definition 2.3.14), coherent (see Definition 2.3.1) and invariant (see Definition 2.3.14).

Lemma 6.2.3. *Let α be an ordinal, X, Y be arbitrary terms, and $x_1, \ldots, x_n \notin \mathrm{FV}(X)$. Then the following conditions hold.*

- *If $X[y/x_1 \ldots x_n] \to^\alpha X'$ then $X' \equiv X''[y/x_1 \ldots x_n]$ where $X[y/Y] \to^\alpha X''[y/Y]$ and $x_1, \ldots, x_n \notin \mathrm{FV}(X'')$.*
- *If $X[y/x_1 \ldots x_n] \succ_i^\alpha d$ then $X[y/Y] \succ_i^\alpha d$.*

Proof. Induction on α. First note that the inductive hypothesis implies:

- if $X[y/x_1 \ldots x_n] \rightsquigarrow_\tau^{<\alpha} d$ then $X[y/Y] \rightsquigarrow_\tau^{<\alpha} d$.

Indeed, assuming $X[y/x_1 \ldots x_n] \stackrel{*}{\to}^{<\alpha} X_1 \succ_\tau^{<\alpha} d$, by the inductive hypothesis there exists X_1' with $x_1, \ldots, x_n \notin \mathrm{FV}(X_1')$ and $X_1 \equiv X_1'[y/x_1 \ldots x_n]$, $X[y/Y] \stackrel{*}{\to}^{<\alpha} X_1'[y/Y]$. Hence $X_1'[y/Y] \succ_\tau^{<\alpha} d$ by applying the IH again. Thus $X[y/Y] \rightsquigarrow_\tau^{<\alpha} d$.

Assume $\Xi(X_1[y/x_1 \ldots x_n])(X_2[y/x_1 \ldots x_n]) \succ_o^\alpha \top$ follows by rule $(\Xi_\top)$, i.e.,

$$X_1[y/x_1 \ldots x_n] \succ_{\mathscr{T}}^{<\alpha} \tau$$

and for every $d \in \mathcal{D}_\tau$ we have $X_2[y/x_1 \ldots x_n]d \rightsquigarrow_o^{<\alpha} \top$. We want to show

$$\Xi(X_1[y/Y])(X_2[y/Y]) \succ_o^\alpha \top.$$

By the IH we obtain $X_1[y/Y] \succ_{\mathscr{T}}^{<\alpha} \tau$ and for every $d \in \mathcal{D}_\tau$ we have $X_2[y/Y] \rightsquigarrow_o^{<\alpha} \top$. Thus $\Xi(X_1[y/Y])(X_2[y/Y]) \succ_o^\alpha \top$ by $(\Xi_\top)$.

Assume $(\lambda u.X_1[y/x_1 \ldots x_n])(X_2[y/x_1 \ldots x_n]) \to_\beta X_1[y/x_1 \ldots x_n][u/X_2[y/x_1 \ldots x_n]]$ and $x_1, \ldots, x_n \notin \mathrm{FV}(X_1, X_2)$. Then $X_1[y/x_1 \ldots x_n][u/X_2[y/x_1 \ldots x_n]] \equiv X_1[u/X_2][y/x_1 \ldots x_n]$. Since $x_1, \ldots, x_n \notin \mathrm{FV}(X_1, X_2)$ then also $x_1, \ldots, x_n \notin \mathrm{FV}(X_1[u/X_2])$. Hence we may take $X'' \equiv X_1[u/X_2]$.

Assume $fX[y/x_1 \ldots x_n] \to^\alpha b$, $f \in \mathcal{D}_{\tau_1\to\tau_2}$, $a \in \mathcal{D}_{\tau_1}$, $b \in \mathcal{D}_{\tau_2}$, $f^{\mathcal{N}}(a) = b$ and $X \succ_{\tau_1}^{<\alpha} a$. Then by the IH we obtain $X[y/Y] \succ_{\tau_1}^{<\alpha}$. Thus also $fX[y/Y] \to^\alpha b$ and we may take $X'' \equiv b$.

Other cases are similar. $\square$

Corollary 6.2.4. *For each ordinal α the reduction system R^α is closed under substitution. In particular, the reduction system R is closed under substitution.*

Lemma 6.2.5. *For $\tau \in \mathscr{T}$, $d, a_1, \ldots, a_n \in \mathbb{T}$ in normal form, and any variable x we have $xa_1 \ldots a_n \not\succ_\tau d$.*

Proof. Induction on the structure of τ. If $xa_1 \ldots a_n \succ_\tau d$ then this may only follow from rule $(\mathcal{F}_\tau)$. Then $\tau = \tau_1 \to \tau_2$, $d \in \mathcal{D}_{\tau_1 \to \tau_2}$, and for $a \in \mathcal{D}_{\tau_1}$ we have $xa_1 \ldots a_n a \leadsto_{\tau_2} d^{\mathcal{N}}(a)$. Since $\mathcal{D}_{\tau_1} \neq \emptyset$, there is $a \in \mathcal{D}_{\tau_1}$ with $xa_1 \ldots a_n a \leadsto_{\tau_2} d^{\mathcal{N}}(a)$, which is only possible when $xa_1 \ldots a_n a \succ_{\tau_2} d^{\mathcal{N}}(a)$. But this is impossible by the inductive hypothesis. □

Lemma 6.2.6. *For all ordinals α, κ the reduction systems R^α and R^κ are mutually coherent. In particular, the reduction system R is coherent.*

Proof. We proceed by induction on pairs of ordinals $\langle \alpha, \kappa \rangle$ ordered componentwise. We need to show the conditions:

(a) $\to^\alpha$ and $\to^\kappa$ commute,

(b) $\to^\kappa$ preserves $\succ_i^\alpha$,

(c) $\to^\alpha$ preserves $\succ_i^\kappa$,

(d) if $X \succ_i^\alpha d_1$ and $X \succ_i^\kappa d_2$ then $d_1 = d_2$,

where $i \in \mathscr{T}$ or $i = \mathscr{T}$.

So assume $(a) - (d)$ hold for all pairs of ordinals $\langle \alpha', \kappa' \rangle$ with $\alpha' < \alpha$ and $\kappa' \leq \kappa$, or $\alpha' \leq \alpha$ and $\kappa' < \kappa$. We show that $(a) - (d)$ also hold for $\langle \alpha, \kappa \rangle$. First we prove the following, for arbitrary terms X, Y.

$(\star)$ If $X \leadsto_i^{<\alpha} d$ and $X \to^\kappa Y$ then $Y \leadsto_i^{<\alpha} d$, where $i \in \mathscr{T}$ or $i = \mathscr{T}$. The same holds with α and κ exchanged.

Assume $X \leadsto_i^{<\alpha} d$ and $X \to^\kappa Y$. Then $X \overset{*}{\to}{}^{<\alpha} X' \succ_i^{<\alpha} d$ for some X'. By part (a) of the IH there is Y' with $Y \overset{*}{\to}{}^{<\alpha} Y'$ and $X' \overset{*}{\to}{}^{\kappa} Y'$. By part (b) of the IH we have $Y' \succ_i^{<\alpha} d$. Thus $Y \leadsto_i^{<\alpha} d$. See Figure 6.2. The proof of the statement with α and κ exchanged is analogous, but using part (c) of the IH instead of part (b).

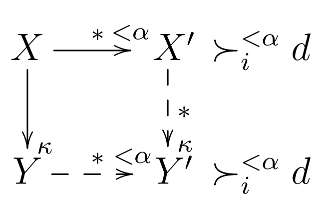

Figure 6.2

We also show the following for arbitrary terms X, Y, and $i \in \mathscr{T}$ or $i = \mathscr{T}$.

$(\star\star)$ If $X \leadsto_i^{<\alpha} d_1$ and $X \leadsto_i^{\kappa} d_2$ then $d_1 = d_2$. The same holds with α and κ exchanged.

Assume $X \rightsquigarrow_i^{<\alpha} d_1$ and $X \rightsquigarrow_i^{\kappa} d_2$. Then there are X_1, X_2 with $X \stackrel{*}{\rightarrow}{}^{<\alpha} X_1 \succ_i^{<\alpha} d_1$ and $X \stackrel{*}{\rightarrow}{}^{\kappa} X_2 \succ_i^{\kappa} d_2$. By part (a) of the IH there is X' with $X_1 \stackrel{*}{\rightarrow}{}^{\kappa} X'$ and $X_2 \stackrel{*}{\rightarrow}{}^{<\alpha} X'$. By parts (b) and (c) of the IH we have $X' \succ_i^{<\alpha} d_1$ and $X' \succ_i^{\kappa} d_2$. By part (d) of the IH we obtain $d_1 = d_2$. See Figure 6.3. The proof for the statement with α and κ exchanged is analogous.

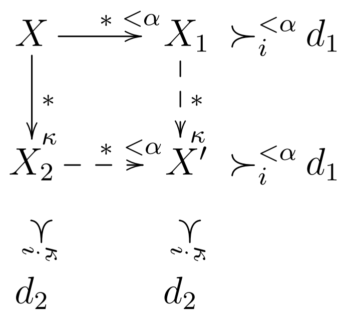

Figure 6.3

Now we prove $(a) - (d)$.

(a) We show that the following pairs of relations commute: $\rightarrow_\gamma^\alpha$ and $\rightarrow_\gamma^\kappa$, $\rightarrow_\gamma^\alpha$ and $\rightarrow_{\beta\eta}$, $\rightarrow_\gamma^\kappa$ and $\rightarrow_{\beta\eta}$. Since $\rightarrow_{\beta\eta}$ is confluent, $\rightarrow^\alpha = \rightarrow_\gamma^\alpha \cup \rightarrow_{\beta\eta}$ and $\rightarrow^\kappa = \rightarrow_\gamma^\kappa \cup \rightarrow_{\beta\eta}$, it then follows from the general Hindley-Rosen Lemma 2.3.3 that $\rightarrow^\alpha$ and $\rightarrow^\kappa$ commute.

Assume $X \rightarrow_\gamma^\alpha X_1$ and $X \rightarrow_\gamma^\kappa X_2$. We show that there is X' with $X_1 \stackrel{\equiv}{\rightarrow}{}_\gamma^\kappa X' \stackrel{\equiv}{\leftarrow}{}_\gamma^\alpha X_2$. Without loss of generality assume that the contraction $X \rightarrow_\gamma^\alpha X_1$ occurs at the root. We have $X \equiv fY$, $f \in \mathcal{D}_{\tau_1 \rightarrow \tau_2}$, $X_1 \equiv f^{\mathcal{N}}(d_1)$ and $Y \succ_{\tau_1}^{<\alpha} d_1$. If the contraction $X \equiv fY \rightarrow_\gamma^\kappa X_2$ also occurs at the root, then $X_2 \equiv f^{\mathcal{N}}(d_2)$, $Y \succ_{\tau_1}^{<\kappa} d_2$ and by part (d) of the IH we obtain $d_1 = d_2$, so we may take $X' \equiv X_1 \equiv X_2$. Otherwise, $X_2 \equiv fY'$ with $Y \rightarrow_\gamma^\kappa Y'$. Since $Y \succ_{\tau_1}^{<\alpha} d_1$, by part (b) of the IH we have $Y' \succ_{\tau_1}^{<\alpha} d_1$. Thus still $X_2 \equiv fY' \rightarrow_\gamma^\alpha d_1 \equiv X_1$, so we may take $X' \equiv X_1$.

It remains to show that $\rightarrow_\gamma^\alpha$ and $\rightarrow_{\beta\eta}$ commute, the proof for $\rightarrow_\gamma^\kappa$ and $\rightarrow_{\beta\eta}$ being analogous. We will show that if $X \rightarrow_\gamma^\alpha X_1$ and $X \rightarrow_{\beta\eta} X_2$ then there is X' such that $X_1 \rightarrow_{\beta\eta} X'$ and $X_2 \stackrel{*}{\rightarrow}{}_\gamma^\alpha X'$. Then the claim will follow by Lemma 2.3.4.

So assume $X \rightarrow_\gamma^\alpha X_1$ and $X \rightarrow_{\beta\eta} X_2$. First suppose the contraction $X \rightarrow_\gamma^\alpha X_1$ is at the root. Then $X \equiv fY$ for some $f \in \mathcal{D}_{\tau_1 \rightarrow \tau_2}$, $Y \succ_{\tau_1}^{<\alpha} d$ and $X_1 \equiv f^{\mathcal{N}}(d)$. Hence $X_2 \equiv fY'$ with $Y \rightarrow_{\beta\eta} Y'$. By part (b) of the IH we obtain $Y' \succ_{\tau_1}^{<\alpha} d$, so still $X_2 \rightarrow_\gamma^\alpha d \equiv X_1$. We may thus take $X' \equiv X_1$.

If the contraction $X \rightarrow_\gamma^\alpha X_1$ is not at the root, then assume without loss of generality that the contraction $X \rightarrow_{\beta\eta} X_2$ is at the root.

Suppose $X \rightarrow_\beta X_2$ occurs at the root. Then $X \equiv (\lambda x.Y_1)Y_2$ and $X_2 \equiv Y_1[x/Y_2]$. If the contraction $X \rightarrow_\gamma^\alpha X_1$ occurs in Y_2, i.e., $Y_2 \rightarrow_\gamma^\alpha Y_2'$ then take $X' \equiv Y_1[x/Y_2']$. We then have $X_2 \equiv Y_1[x/Y_2] \stackrel{*}{\rightarrow}{}_\gamma^\alpha Y_1[x/Y_2'] \equiv X'$, and

$$X_1 \equiv (\lambda x.Y_1)Y_2' \rightarrow_\beta Y_1[x/Y_2']$$

as required. Otherwise, the contraction $X \to_\gamma^\alpha X_1$ occurs in Y_1, i.e., $Y_1 \to_\gamma^\alpha Y_1'$. Then $Y_1[x/Y_2] \to_\gamma^\alpha Y_1'[x/Y_2]$ by Corollary 6.2.4, and we may take $X' \equiv Y_1'[x/Y_2]$.

Finally, suppose $X \to_\eta X_2$ occurs at the root. Then $X \equiv \lambda x.X_2 x$ with $x \notin \mathrm{FV}(X_2)$. It is impossible that $X_2 x$ is the redex contracted in $X \to_\gamma^\alpha X_1$. Indeed, otherwise $x \succ_\tau d$ for some $d \in \mathcal{D}_\tau$, which is impossible by Lemma 6.2.5. So the contraction $X \to_\gamma^\alpha X_1$ must occur inside X_2, i.e., $X_2 \to_\gamma^\alpha X_2'$. Then we may simply take $X' \equiv X_2'$.

(b) Assume $X \succ_i^\alpha d$ and $X \to^\kappa X'$. We need to show $X' \succ_i^\alpha d$. We consider possible cases according to the definition of $X \succ_i^\alpha d$.

Assume $X \succ_i^\alpha d$ follows from $(\Xi_\top)$, i.e., $i = o$, $X \equiv \Xi X_1 X_2$, $d \equiv \top$, $X_1 \succ_{\mathscr{T}}^{<\alpha} \tau$ for some $\tau \in \mathscr{T}$, and for every $a \in \mathcal{D}_\tau$ we have $X_2 a \rightsquigarrow_o^{<\alpha} \top$. Then also $X' \equiv \Xi X_1' X_2'$ with $X_k \overset{\equiv}{\to}{}^\kappa X_k'$. By part ($b$) of the IH we have $X_1' \succ_{\mathscr{T}}^{<\alpha} \tau$. By ($\star$), for every $a \in \mathcal{D}_\tau$ we have $X_2' a \rightsquigarrow_o^{<\alpha} \top$. Hence $X' \equiv \Xi X_1' X_2' \succ_o^\alpha \top$ by $(\Xi_\top)$.

Assume $X \succ_i^\alpha d$ follows from $(\Xi_\bot)$, i.e., $i = o$, $X \equiv \Xi X_1 X_2$, $d \equiv \bot$, $X_1 \succ_{\mathscr{T}}^{<\alpha} \tau$ for some $\tau \in \mathscr{T}$, and there exists $a \in \mathcal{D}_\tau$ with $X_2 a \rightsquigarrow_o^{<\alpha} \bot$. Then also $X' \equiv \Xi X_1' X_2'$ with $X_k \overset{\equiv}{\to}{}^\kappa X_k'$. By part ($b$) of the IH we have $X_1' \succ_{\mathscr{T}}^{<\alpha} \tau$. By ($\star$) we also have $X_2' a \rightsquigarrow_o^{<\alpha} \bot$. Hence $X' \equiv \Xi X_1' X_2' \succ_o^\alpha \bot$ by $(\Xi_\bot)$.

Assume $X \succ_i^\alpha d$ follows from $(\mathsf{F}_{\mathscr{T}})$, i.e., $i = \mathscr{T}$, $X \equiv \lambda f.\Xi X_1(\lambda x.Y[y/fx])$, $d = \tau_1 \to \tau_2 \in \mathscr{T}$, $f, x \notin \mathrm{FV}(X, Y)$ $X_1 \succ_{\mathscr{T}}^{<\alpha} \tau_1$ and $\lambda y.Y \rightsquigarrow_{\mathscr{T}}^{<\alpha} \tau_2$. If the contraction $X \to^\kappa X'$ occurs inside X_1 then it follows from the IH and $(\mathsf{F}_{\mathscr{T}})$ that $X' \succ_{\mathscr{T}}^\alpha \tau$. Otherwise the contraction occurs in $\lambda x.Y[y/fx]$, i.e., $\lambda x.Y[y/fx] \to^\kappa Z$. If $\lambda x.Y[y/fx] \equiv \lambda x.fx \to_\eta f \equiv Z$ then $Y \equiv y$ and $\lambda y.y \succ_{\mathscr{T}}^{<\alpha} \tau_2$, which is impossible by Definition 6.2.1. Hence by Lemma 6.2.3 we have $Y' \equiv Y''[y/fx]$ with $Y \to^\kappa Y''$ and $f, x \notin \mathrm{FV}(Y'')$. Thus $\lambda y.Y \to^\kappa \lambda y.Y''$, so $\lambda y.Y'' \rightsquigarrow_{\mathscr{T}}^{<\alpha} \tau_2$ by ($\star$). Then $Y' \succ_{\mathscr{T}}^\alpha \tau$ follows from $(\mathsf{F}_{\mathscr{T}})$.

Assume $X \succ_i^\alpha d$ follows from $(\mathcal{F}_\tau)$, i.e., $i = \tau_1 \to \tau_2$, $d \in \mathcal{D}_{\tau_1 \to \tau_2}$, and for every $a \in \mathcal{D}_{\tau_1}$ we have $Xa \rightsquigarrow_{\tau_2}^{<\alpha} d^{\mathcal{N}}(a)$. By ($\star$), for $a \in \mathcal{D}_{\tau_1}$ we have $X'a \rightsquigarrow_{\tau_2}^{<\alpha} d^{\mathcal{N}}(a)$. Thus $X' \succ_i^\alpha d$ by $(\mathcal{F}_\tau)$.

Assume $X \succ_i^\alpha d$ follows from $(\neg_\top)$, i.e., $i = o$, $d \equiv \top$, $X \equiv \neg Y$ and $Y \succ_o^{<\alpha} \bot$. Then $X' \equiv \neg Y'$ with $Y \to^\kappa Y'$. We have $Y' \succ_o^{<\alpha} \bot$ by part (b) of the IH. Thus $X' \succ_o^\alpha \top$ by $(\neg_\top)$.

Other cases are similar.

(c) Analogous to (b).

(d) Suppose $X \succ_i^\alpha d_1$ and $X \succ_i^\kappa d_2$. We need to show $d_1 = d_2$. We consider all possible overlaps of rules in Definition 6.2.1, i.e., all possible pairs of rules by which $X \succ_i^\alpha d_1$ and $X \succ_i^\kappa d_2$ could be obtained.

Assume both $X \succ_i^\alpha d_1$ and $X \succ_i^\kappa d_2$ follow from $(\mathcal{F}_\tau)$. Then $i = \tau = \tau_1 \to \tau_2$, $d_1, d_2 \in \mathcal{D}_\tau$ and for $a \in \mathcal{D}_{\tau_1}$ we have $Xa \rightsquigarrow_{\tau_2}^{<\alpha} d_1^{\mathcal{N}}(a)$ and $Xa \rightsquigarrow_{\tau_2}^{<\kappa} d_2^{\mathcal{N}}(a)$. Then $d_1^{\mathcal{N}}(a) = d_2^{\mathcal{N}}(a)$ for $a \in \mathcal{D}_{\tau_1}$, by ($\star\star$). Thus $d_1 = d_2$.

Assume $X \succ_i^\alpha d_1$ follows from $(\Xi_\top)$ and $X \succ_i^\kappa d_2$ from $(\Xi_\bot)$. Then $i = o$, $X \equiv \Xi X_1 X_2$, $d_1 \equiv \top$, $d_2 \equiv \bot$ and

– $X_1 \succ_{\mathscr{T}}^{<\alpha} \tau$ and for all $d \in \mathcal{D}_\tau$ we have $X_2 d \rightsquigarrow_o^{<\alpha} \top$, and
– $X_1 \succ_{\mathscr{T}}^{<\kappa} \tau'$ and there is $d' \in \mathcal{D}_{\tau'}$ with $X_2 d' \rightsquigarrow_o^{<\kappa} \bot$.

By part (d) of the IH we have $\tau = \tau'$. But then $X_2 d' \rightsquigarrow_o^{<\alpha} \top$ and $X_2 d' \rightsquigarrow_o^{<\kappa} \bot$. This contradicts $(\star\star)$.

Assume $X \succ_i^\alpha d_1$ follows from $(\neg_\top)$ and $X \succ_i^\kappa d_2$ follows from $(\neg_\bot)$. Then $i = o$, $d_1 \equiv \top$, $d_2 \equiv \bot$, $X \equiv \neg Y$, $Y \succ_o^{<\alpha} \bot$ and $Y \succ_o^{<\kappa} \top$. But $Y \succ_o^{<\alpha} \bot$ and $Y \succ_o^{<\kappa} \top$ cannot both hold by part (d) of the IH.

Other cases are similar.

□

Definition 6.2.7. The *rank* of a type $\tau \in \mathscr{T}$, denoted $\mathrm{rank}(\tau)$, is defined as follows. If $\tau \in \mathcal{B} \cup \{o\}$ then $\mathrm{rank}(\tau) = 1$. Otherwise $\tau = \tau_1 \to \tau_2$ and we set

$$\mathrm{rank}(\tau) = \max\{\mathrm{rank}(\tau_1) + 1, \mathrm{rank}(\tau_2)\}.$$

We write $X \gg^n Y$ if there exists a term Z, distinct variables $x_1, \ldots, x_m \in \mathrm{FV}(X)$, and terms $X_1, \ldots, X_m$, $d_1, \ldots, d_m$ such that:

- $X \equiv Z[x_1/X_1, \ldots, x_m/X_m]$,
- $Y \equiv Z[x_1/d_1, \ldots, x_m/d_m]$,
- for each $k = 1, \ldots, m$ there is $\tau \in \mathscr{T}$ with $\mathrm{rank}(\tau) \leq n$ and $X_k \succ_\tau d_k$.

We set $\gg^{<n} = \bigcup_{m<n} \gg^m$ and $\gg = \bigcup_{n \in \mathbb{N}} \gg^n$.

The following simple lemma will be used implicitly.

Lemma 6.2.8.

1. *If $X \gg^n Y_1 Y_2$ then $X \equiv X_1 X_2$ with $X_1 \gg^n Y_1$ and $X_2 \gg^n Y_2$.*
2. *If $X \gg^n \lambda x.Y$ then $X \equiv \lambda x.X'$ with $X' \gg^n Y$. Moreover, if $X_1, \ldots, X_m$ are as in the definition of $X' \gg^n Y$, then $x \notin \mathrm{FV}(X_1, \ldots, X_m)$.*

Proof. Follows directly from Definition 6.2.7. □

Lemma 6.2.9. *The reduction system R is invariant.*

Proof. We show the following two conditions by induction on pairs $\langle n, \alpha \rangle$ ordered lexicographically, i.e., $\langle n_1, \alpha_1 \rangle < \langle n_2, \alpha_2 \rangle$ iff $n_1 < n_2$, or $n_1 = n_2$ and $\alpha_1 < \alpha_2$.

(1) If $X \gg^n Y \succ_i^\alpha d$ then $X \succ_i d$, where $i \in \mathscr{T}$ or $i = \mathscr{T}$.

(2) If $X \gg^n Y \xrightarrow{*}^{\alpha} Z$ then there is Y' with $X \xrightarrow{*} Y' \gg^n Z$.

For $\alpha = \zeta$, where ζ is the closure ordinal of Definition 6.2.1, the above conditions imply the invariance of R. Indeed, assuming $X \succ_i d$ and $Yd \rightsquigarrow_j d'$, we have $YX \gg Yd \xrightarrow{*} \cdot \succ_j d'$, so $YX \xrightarrow{*} \cdot \gg \cdot \succ_j d'$ by (2), hence $YX \xrightarrow{*} \cdot \succ_j d'$ by (1), and thus $YX \rightsquigarrow_j d'$.

So assume (1) and (2) hold for all $\langle n', \alpha' \rangle < \langle n, \alpha \rangle$. First we show the following:

($\star$) if $X \gg^n Y \rightsquigarrow_\tau^{<\alpha} d$ then $X \rightsquigarrow_\tau d$.

Assume $X \gg^n Y \xrightarrow{*}^{<\alpha} \cdot \succ_\tau^{<\alpha} d$. Applying part (2) of the IH we obtain $X \xrightarrow{*} \cdot \gg^n \cdot \succ_\tau^{<\alpha} d$, and thus $X \xrightarrow{*} \cdot \succ_\tau d$ by part (1) of the IH, i.e., $X \rightsquigarrow_\tau d$.

Now we show (1) and (2) for $\langle n, \alpha \rangle$.

(1) Assume $X \gg^n Y \succ_i^\alpha d$ where $i \in \mathscr{T}$ or $i = \mathscr{T}$. We consider all possible rules by which $Y \succ_i^\alpha d$ could be obtained.

($\mathcal{D}_\tau$) Then $X \gg^n d \succ_\tau^\alpha d$. This is only possible when $X \equiv d$ or $X \succ_\tau d$. In any case $X \succ_\tau d$.

($\mathcal{F}_\tau$) Then $X \gg^n Y \succ_\tau^\alpha d$, $\tau = \tau_1 \to \tau_2$, $d \in \mathcal{D}_\tau$, and for every $a \in \mathcal{D}_{\tau_1}$ we have $Ya \rightsquigarrow_{\tau_2}^{<\alpha} d^{\mathcal{N}}(a)$. Let $a \in \mathcal{D}_{\tau_1}$. Then $Xa \gg^n Ya \rightsquigarrow_{\tau_2}^{<\alpha} d^{\mathcal{N}}(a)$. Thus $Xa \rightsquigarrow_{\tau_2} d^{\mathcal{N}}(a)$ by ($\star$). Since $a \in \mathcal{D}_{\tau_1}$ was arbitrary, we conclude $X \succ_\tau d$.

($\neg_\top$) Then $X \gg^n \neg Y' \succ_o^\alpha \top$ and $Y' \succ_o^{<\alpha} \bot$. We have $X \equiv \neg X'$ with $X' \gg^n Y'$. So $X' \succ_o \bot$ by the IH. Therefore $X \equiv \neg X' \succ_o \top$ by ($\neg_\top$).

($\Xi_\top$) Then $X \gg^n \Xi Y_1 Y_2 \succ_o^\alpha \top$, $Y_1 \succ_{\mathscr{T}}^{<\alpha} \tau$ and for every $d \in \mathcal{D}_\tau$ we have $Y_2 d \rightsquigarrow_o^{<\alpha} \top$. We have $X \equiv \Xi X_1 X_2$ with $X_k \gg^n Y_k$. So $X_1 \gg^n Y_1 \succ_{\mathscr{T}}^{<\alpha} \tau$, and by part (1) of the IH we obtain $X_1 \succ_{\mathscr{T}} \tau$. If $d \in \mathcal{D}_\tau$ then $X_2 d \gg^n Y_2 d \rightsquigarrow_o^{<\alpha} \top$, so $X_2 d \rightsquigarrow_o \top$ by ($\star$). Therefore, $X \equiv \Xi X_1 X_2 \succ_o \top$.

($\Xi_\bot$) Then $X \gg^n \Xi Y_1 Y_2 \succ_o^\alpha \bot$, $Y_1 \succ_{\mathscr{T}}^{<\alpha} \tau$ and there is $d \in \mathcal{D}_\tau$ with $Y_2 d \rightsquigarrow_o^{<\alpha} \bot$. We have $X \equiv \Xi X_1 X_2$ with $X_k \gg^n Y_k$. So $X_1 \gg^n Y_1 \succ_{\mathscr{T}}^{<\alpha} \tau$, and by part (1) of the IH we obtain $X_1 \succ_{\mathscr{T}} \tau$. Also $X_2 d \gg^n Y_2 d \rightsquigarrow_o^{<\alpha} \bot$, so $X_2 d \rightsquigarrow_o \bot$ by ($\star$). Therefore, $X \equiv \Xi X_1 X_2 \succ_o \bot$.

($\mathsf{A}_\top$) Then $X \gg^n \mathsf{A}_\tau d \succ_o \top$ with $\tau \in \mathcal{B}$ and $d \in \mathcal{D}_\tau$. Because $\tau \in \mathcal{B}$ we must have $X \equiv \mathsf{A}_\tau d$. Indeed, the only other possibility would be $X \equiv \mathsf{A}_\tau X'$ with $X' \succ_\tau d$, but by inspecting Definition 6.2.1 one sees that for $\tau \in \mathcal{B}$ this implies $X' \equiv d$.

($\mathsf{F}_{\mathscr{T}}$) Then $X \gg^n \lambda f.\Xi Y_1(\lambda x.Y_2[y/fx]) \succ_{\mathscr{T}}^\alpha \tau_1 \to \tau_2$, $f, x \notin \mathrm{FV}(Y_1, Y_2)$, $Y_1 \succ_{\mathscr{T}}^{<\alpha} \tau_1$ and $\lambda y.Y_2 \rightsquigarrow_{\mathscr{T}}^{<\alpha} \tau_2$. We have $X \equiv \lambda f.\Xi X_1(\lambda x.X_2[y/fx])$ with $f, x \notin \mathrm{FV}(X_1, X_2)$, $X_1 \gg^n Y_1$ and $X_2 \gg^n Y_2$ (because if $Z \succ_j d$ then d is closed and in particular $f, x \notin \mathrm{FV}(d)$, so $M \gg^n Y_2[y/fx]$ implies $M \equiv X_2[y/fx]$ with $X_2 \gg^n Y_2$). Thus $X_1 \gg^n Y_1 \succ_{\mathscr{T}}^{<\alpha} \tau_1$ and $\lambda y.X_2 \gg^n \lambda y.Y_2 \rightsquigarrow_{\mathscr{T}}^{<\alpha} \tau_2$. Hence $X_1 \succ_{\mathscr{T}} \tau_1$ by part (1) of the IH. Also $\lambda y.X_2 \rightsquigarrow_{\mathscr{T}} \tau_2$ by ($\star$). Thus $X \succ_{\mathscr{T}} \tau_1 \to \tau_2$.

Other cases are similar.

(2) It suffices to show that if $X \gg^n Y \to^\alpha Z$ then $X \xrightarrow{*} \cdot \gg^n Z$. Without loss of generality, we may assume that the contraction $Y \to^\alpha Z$ occurs at the root. We consider possible rules by which this contraction could occur.

(β) Then $Y \equiv (\lambda x.Y_1)Y_2$, $X \equiv (\lambda x.X_1)X_2$, $Z \equiv Y_1[x/Y_2]$ and $X_k \gg^n Y_k$. Note that $X_1[x/X_2] \gg^n Y_1[x/Y_2]$ follows from Lemma 6.2.8. Indeed,

$$X_k \equiv X'_k[x_1/M_1, \ldots, x_m/M_m]$$

and

$$Y_k \equiv Y'_k[x_1/d_1, \ldots, x_m/d_m]$$

with $M_i \succ_{\tau_i} d_i$ and $x \notin \mathrm{FV}(M_1, \ldots, M_m)$, so

$$(X'_1[x_1/M_1, \ldots, x_m/M_m])[x/X'_2[x_1/M_1, \ldots, x_m/M_m]] \equiv X'_1[x/X'_2][x_1/M_1, \ldots, x_m/M_m].$$

Also

$$(Y'_1[x_1/d_1, \ldots, x_m/d_m])[x/Y'_2[x_1/d_1, \ldots, x_m/d_m]] \equiv Y'_1[x/Y'_2][x_1/d_1, \ldots, x_m/d_m].$$

Hence

$$X_1[x/X_2] \equiv X'_1[x/X'_2][x_1/M_1, \ldots, x_m/M_m]$$

and

$$Y_1[x/Y_2] \equiv Y'_1[x/Y'_2][x_1/d_1, \ldots, x_m/d_m]$$

so $X \to_\beta X_1[x/X_2] \gg^n Y_1[x/Y_2] \equiv Z$.

(η) Then $Y \equiv \lambda x.Zx$ and $X \equiv \lambda x.X'x$ with $X' \gg^n Z$, $n \notin \mathrm{FV}(X', Z)$. Therefore $X \to_\eta X' \gg^n Z$.

(γ) We have $X \gg^n Y \to^\alpha_\gamma Z$. There are two possibilities.

1. $X \equiv fX'$, $Y \equiv fY'$, $X' \gg^n Y'$ and $f \in \mathcal{D}_{\tau_1 \to \tau_2}$. Then $Y' \succ^{<\alpha}_{\tau_1} d$ for some $d \in \mathcal{D}_{\tau_1}$ and $Z \equiv f^{\mathcal{N}}(d)$. By part (1) of the IH we have $X' \succ_{\tau_1} d$. So $X \equiv fX' \to_\gamma f^{\mathcal{N}}(d) \equiv Z \gg^n Z$.
2. $X \equiv FX'$, $Y \equiv fY'$, $F \succ_\tau f$, $X' \gg^n Y'$, $f \in \mathcal{D}_\tau$, $\mathrm{rank}(\tau) \leq n$ and $\tau = \tau_1 \to \tau_2$. Then also $Y' \succ^{<\alpha}_{\tau_1} d$ for some $d \in \mathcal{D}_{\tau_1}$ and $Z \equiv f^{\mathcal{N}}(d)$. Since $X' \gg^n Y' \succ^{<\alpha}_{\tau_1} d$, we have $X' \succ_{\tau_1} d$ by part (1) of the IH. Since $\tau = \tau_1 \to \tau_1$ one sees by inspecting Definition 6.2.1 that $F \succ_\tau f$ can only be obtained by rule ($\mathcal{F}_\tau$). Since $d \in \mathcal{D}_{\tau_1}$ we thus have $Fd \rightsquigarrow_{\tau_2} f^{\mathcal{N}}(d) \equiv Z$. We have $FX' \gg^{<n} Fd \xrightarrow{*} \cdot \succ_{\tau_2} Z$, because $\mathrm{rank}(\tau_1) < \mathrm{rank}(\tau) \leq n$ and $X' \succ_{\tau_1} d$. Thus $FX' \xrightarrow{*} \cdot \gg^{<n} \cdot \succ_{\tau_2} Z$ by part (2) of the IH. So $FX' \xrightarrow{*} \cdot \succ_{\tau_2} Z$ by part (1) of the IH. Since $\mathrm{rank}(\tau_2) \leq \mathrm{rank}(\tau) \leq n$, $X \equiv FX' \xrightarrow{*} \cdot \gg^n Z$.

□

Coherence of the system R together with Lemma 2.3.5 and Lemma 2.3.6 implies the following for $i \in \mathscr{T}$ or $i = \mathscr{T}$:

- $X \rightsquigarrow_i d$ iff $X =_R \cdot \succ_i d$,
- if $X \rightsquigarrow_i d_1$ and $X \rightsquigarrow_i d_2$ then $d_1 \equiv d_2$.

In particular, if $X \rightsquigarrow_i d$ and $X =_R Y$ then also $Y \rightsquigarrow_i d$.

Coherence and invariance also imply that if $X \rightsquigarrow_i d$ and $Y[x/d] \rightsquigarrow_i d'$ then $Y[x/X] \rightsquigarrow_i d'$. Indeed, assume $X \rightsquigarrow_i d$ and $Y[x/d] \rightsquigarrow_i d'$. Then $(\lambda x.Y)d \to Y[x/d] \rightsquigarrow_i d'$, so $(\lambda x.Y)d \rightsquigarrow_i d'$. By Lemma 2.3.15 we obtain $(\lambda x.Y)X \rightsquigarrow_i d'$. Since $(\lambda x.Y)X \to Y[x/X]$, by coherence we have $Y[x/X] \rightsquigarrow_i d'$.

In what follows we use the above simple properties implicitly, only noting that something follows by coherence and/or invariance (of the system R).

Also note that if, e.g., $\Xi XY \rightsquigarrow_o \top$ then $X \rightsquigarrow_{\mathscr{T}} \tau$ for some $\tau \in \mathscr{T}$ and for all $d \in \mathcal{D}_\tau$ we have $Yd \rightsquigarrow_o \top$. Indeed, $\Xi XY \xrightarrow{*} Z \succ_o \top$ implies $Z \equiv \Xi X'Y'$ with $X \xrightarrow{*} X'$ and $Y \xrightarrow{*} Y'$. By inspecting Definition 6.2.1 one sees that $\Xi X'Y' \succ_o \top$ can only be obtained by rule $(\Xi_\top)$. Hence $X \xrightarrow{*} X' \succ_{\mathscr{T}} \tau$ for some $\tau \in \mathscr{T}$ and for all $d \in \mathcal{D}_\tau$ we have $Yd \xrightarrow{*} Y'd \rightsquigarrow_o \top$.

For the sake of brevity, justifications of analogous trivial observations will be left implicit.

Lemma 6.2.10. *If $X \succ_{\mathscr{T}} \tau$ then for any Z with $XZ \rightsquigarrow_o \top$ there is $d \in \mathcal{D}_\tau$ with $Z \rightsquigarrow_\tau d$.*

Proof. We proceed by induction on the structure of τ. Suppose $X \succ_{\mathscr{T}} \tau$ and $XZ \rightsquigarrow_o \top$.

If $\tau \in \mathcal{B}$ then $X \equiv \mathsf{A}_\tau$ and $\mathsf{A}_\tau Z \xrightarrow{*} \mathsf{A}_\tau Z' \succ_o \top$ where $Z \xrightarrow{*} Z'$. Then $Z' \equiv d$ for some $d \in \mathcal{D}_\tau$, so $Z \rightsquigarrow_\tau d$. If $\tau = o$ then $X \equiv \mathsf{H}$, and $\mathsf{H}Z \rightsquigarrow_o \top$. By coherence (Lemma 6.2.6) we have $Z \vee \neg Z \rightsquigarrow_o \top$. This implies $Z \rightsquigarrow_o \top$ or $Z \rightsquigarrow_o \bot$, and we are done because $\top, \bot \in \mathcal{D}_o$.

So assume $\tau = \tau_1 \to \tau_2$. Then $X \equiv \lambda f.\Xi X_1(\lambda x.X_2[y/fx])$ with $f, x \notin \mathrm{FV}(X_1, X_2)$, $X_1 \succ_{\mathscr{T}} \tau_1$ and $\lambda y.X_2 \succ_{\mathscr{T}} \tau_2$. Since $XZ \rightsquigarrow_o \top$, by coherence $\Xi X_1(\lambda x.X_2[y/Zx]) \rightsquigarrow_o \top$. By coherence and $(\Xi_\top)$ this implies that for every $d \in \mathcal{D}_{\tau_1}$ we have $(\lambda y.X_2)(Zd) \rightsquigarrow_o \top$. Since $\lambda y.X_2 \succ_{\mathscr{T}} \tau_2$, by the IH, for every $d \in \mathcal{D}_{\tau_1}$ there is $a_d \in \mathcal{D}_{\tau_2}$ with $Zd \rightsquigarrow_{\tau_2} a_d$. So by $(\mathcal{F}_\tau)$ we have $Z \succ_\tau f$ for $f \in \mathcal{D}_\tau$ such that $f^{\mathcal{N}}(d) = a_d$ for $d \in \mathcal{D}_{\tau_1}$. □

Lemma 6.2.11. *If $d \in \mathcal{D}_\tau$ for $\tau \in \mathscr{T}$, then $d \succ_\tau d$.*

Proof. Induction on the size of τ. □

Lemma 6.2.12. *If $X \succ_{\mathscr{T}} \tau$ then $Xd \rightsquigarrow_o \top$ for any $d \in \mathcal{D}_\tau$.*

Proof. Induction on the structure of τ. Suppose $X \succ_{\mathscr{T}} \tau$ and $d \in \mathcal{D}_\tau$.

If $\tau \in \mathcal{B}$ then $X \equiv \mathsf{A}_\tau$ and $\mathsf{A}_\tau d \succ_o \top$ by $(\mathsf{A}_\top)$. If $\tau = o$ then $X \equiv \mathsf{H}$, $d \in \{\top, \bot\}$, and $\mathsf{H}d \succ_o \top$ follows from definitions.

So assume $\tau = \tau_1 \to \tau_2$. Then $X =_\beta \mathsf{F}X_1X_2$ with $X_1 \succ_{\mathscr{T}} \tau_1$ and $X_2 \succ_{\mathscr{T}} \tau_2$. Let $a \in \mathcal{D}_{\tau_1}$. Then $X_2(d^{\mathcal{N}}(a)) \rightsquigarrow_o \top$ by the IH. By Lemma 6.2.11 we have $a \succ_{\tau_1} a$, so $da \to_\gamma d^{\mathcal{N}}(a)$. Hence $X_2(da) \rightsquigarrow_o \top$. Thus $Xd \rightsquigarrow_o \top$ by $(\Xi_\top)$ and coherence. □

Lemma 6.2.13. *The following conditions hold.*

1. *$\Xi XY \rightsquigarrow_o \top$ iff $\mathsf{L}X \rightsquigarrow_o \top$ and for every Z with $XZ \rightsquigarrow_o \top$ we have $YZ \rightsquigarrow_o \top$.*
2. *$\Xi XY \rightsquigarrow_o \bot$ iff $\mathsf{L}X \rightsquigarrow_o \top$ and there exists Z with $XZ \rightsquigarrow_o \top$ and $YZ \rightsquigarrow_o \bot$.*

Proof. Follows from Lemma 6.2.6, Lemma 6.2.9, Lemma 6.2.10 and Lemma 6.2.12. □

Lemma 6.2.14. *If $X \rightsquigarrow_{\mathscr{T}} \tau_1$ and $Y \rightsquigarrow_{\mathscr{T}} \tau_2$ then $\mathsf{F}XY \rightsquigarrow_{\mathscr{T}} \tau_1 \to \tau_2$.*

Proof. We have $\mathsf{F}XY \equiv \lambda f.\Xi X(\lambda x.Y(fx))$ with $f, x \notin \mathrm{FV}(X, Y)$. Assume $X \rightsquigarrow_{\mathscr{T}} \tau_1$, i.e., $X \xrightarrow{*} X' \succ_{\mathscr{T}} \tau_1$, and assume $Y \rightsquigarrow_{\mathscr{T}} \tau_2$. Then $\lambda y.Yy \rightsquigarrow_{\mathscr{T}} \tau_2$ for $y \notin \mathrm{FV}(Y)$. By $(\mathsf{F}_{\mathscr{T}})$ this implies $\mathsf{F}XY \xrightarrow{*} \lambda f.\Xi X'(\lambda x.Y(fx)) \succ_{\mathscr{T}} \tau_1 \to \tau_2$ where $f, x \notin \mathrm{FV}(X, Y)$. Hence $\mathsf{F}XY \rightsquigarrow_{\mathscr{T}} \tau_1 \to \tau_2$. □

Lemma 6.2.15. *If* $\mathsf{L}X \rightsquigarrow_o \top$*, and either* $\mathsf{L}Y \rightsquigarrow_o \top$ *or there is no* Z *with* $XZ \rightsquigarrow_o \top$*, then* $\mathsf{L}(\mathsf{F}XY) \rightsquigarrow_o \top$*.*

Proof. Assume the antecedent of the implication in the lemma. Since $\mathsf{L}X \rightsquigarrow_o \top$, there is $\tau_1 \in \mathscr{T}$ with $X \rightsquigarrow_{\mathscr{T}} \tau_1$. We have $\mathcal{D}_{\tau_1} \neq \emptyset$, so by Lemma 6.2.12 there is $d \in \mathcal{D}_{\tau_1}$ with $Xd \rightsquigarrow_o \top$. Hence $\mathsf{L}Y \rightsquigarrow_o \top$, so there is $\tau_2 \in \mathscr{T}$ with $Y \rightsquigarrow_{\mathscr{T}} \tau_2$. Thus $\mathsf{F}XY \rightsquigarrow_{\mathscr{T}} \tau_1 \to \tau_2$ by Lemma 6.2.14. Hence $\mathsf{L}(\mathsf{F}XY) \rightsquigarrow_o \top$. □

The model we construct will in fact be an $e\mathcal{I}\mathrm{K}\omega$-model, validating extensionality of Leibniz equality. To show this we need the following lemmas.

Lemma 6.2.16. *If* $p \in \mathcal{D}_{\tau_1 \to \tau_2}$ *and* $pX \rightsquigarrow_{\tau_2} b$ *for some* $b \in \mathcal{D}_{\tau_2}$*, then there is* $a \in \mathcal{D}_{\tau_1}$ *with* $X \rightsquigarrow_{\tau_1} a$ *and* $p^{\mathcal{N}}(a) \equiv b$*.*

Proof. Assume $p \in \mathcal{D}_{\tau_1 \to \tau_2}$ and $pX \rightsquigarrow_{\tau_2} b$ for some $b \in \mathcal{D}_{\tau_2}$. By straightforward induction on α one shows

- ($\star$) for any $n \in \mathbb{N}$, any terms $X_1, \ldots, X_n, c$ and any type $\tau \in \mathcal{T}$, if $pX_1 \ldots X_n \rightsquigarrow_\tau c$ then there are a term X' and a constant d such that $X_1 \xrightarrow{*} X'$ and $pX' \to_\gamma d$.

Using ($\star$) and coherence we conclude that there is X' with $X \xrightarrow{*} X'$ and $pX' \to_\gamma b$. But then $X' \succ_{\tau_1} a$ for $a \in \mathcal{D}_{\tau_1}$ such that $p^{\mathcal{N}}(a) \equiv b$. So $X \rightsquigarrow_{\tau_1} a$. □

Lemma 6.2.17. *If* $\mathsf{Q}_L AXY \rightsquigarrow_o \top$ *and* $A \rightsquigarrow_{\mathscr{T}} \tau$ *then there is* $d \in \mathcal{D}_\tau$ *such that* $X \rightsquigarrow_\tau d$ *and* $Y \rightsquigarrow_\tau d$*.*

Proof. Recall that $\mathsf{Q}_L AXY = \Xi(\mathsf{F}A\mathsf{H})(\lambda p. \neg(pX) \vee pY)$. Assume $\mathsf{Q}_L AXY \rightsquigarrow_o \top$ and $A \rightsquigarrow_{\mathscr{T}} \tau$. Then $\mathsf{F}A\mathsf{H} \rightsquigarrow_{\mathscr{T}} \tau \to o$. Let $p \in \mathcal{D}_{\tau \to o}$ be such that $p^{\mathcal{N}}(d) \equiv \bot$ for $d \in \mathcal{D}_\tau$. We have $\neg(pX) \vee pY \rightsquigarrow_o \top$, so $\neg(pX) \rightsquigarrow_o \top$ or $pY \rightsquigarrow_o \top$. If $pY \rightsquigarrow_o \top$ then by Lemma 6.2.16 there is $d \in \mathcal{D}_\tau$ with $p^{\mathcal{N}}(d) \equiv \top$, which contradicts $p^{\mathcal{N}}(d) \equiv \bot$. Hence $\neg(pX) \rightsquigarrow_o \top$, so $pX \rightsquigarrow_o \bot$. By Lemma 6.2.16 there is $d_X \in \mathcal{D}_\tau$ with $X \rightsquigarrow_\tau d_X$. By an analogous argument, using $p \in \mathcal{D}_{\tau \to o}$ such that $p^{\mathcal{N}}(d) \equiv \top$ for $d \in \mathcal{D}_\tau$, one concludes that there is d_Y with $Y \rightsquigarrow_\tau d_Y$. Suppose $d_X \not\equiv d_Y$. Take $p \in \mathcal{D}_{\tau \to o}$ such that $p^{\mathcal{N}}(d_X) \equiv \top$ and $p^{\mathcal{N}}(d_Y) \equiv \bot$. We have $\neg(pd_X) \vee pd_Y \rightsquigarrow_o \bot$, so $\neg(pX) \vee pY \rightsquigarrow_o \bot$ by invariance. But this contradicts $\mathsf{Q}_L AXY \rightsquigarrow_o \top$. □

Lemma 6.2.18. *If* $\mathsf{L}A \rightsquigarrow_o \top$ *and for every* Z *with* $AZ \rightsquigarrow_o \top$ *we have* $\mathsf{Q}_L B(XZ)(YZ) \rightsquigarrow_o \top$*, then* $\mathsf{Q}_L(\mathsf{F}AB)XY \rightsquigarrow_o \top$*.*

Proof. Recall that $\mathsf{Q}_L AXY =_\beta \Xi(\mathsf{F}A\mathsf{H})(\lambda p. \neg(pX) \vee pY)$.

Suppose $\mathsf{L}A \rightsquigarrow_o \top$ and for every Z with $AZ \rightsquigarrow_o \top$ we have $\mathsf{Q}_L B(XZ)(YZ) \rightsquigarrow_o \top$. Since $\mathsf{L}A \rightsquigarrow_o \top$, we have $A \rightsquigarrow_{\mathscr{T}} \tau_1$ for some $\tau_1 \in \mathscr{T}$ by ($\mathsf{L}_\top$) in Definition 6.2.1. Because $\mathcal{D}_{\tau_1} \neq \emptyset$, there is $d \in \mathcal{D}_{\tau_1}$, and by Lemma 6.2.12 we have $Ad \rightsquigarrow_o \top$. Thus $\mathsf{Q}_L B(Xd)(Yd) \rightsquigarrow_o \top$, so $\mathsf{L}B \rightsquigarrow_o \top$ by ($\mathsf{L}_\top$), ($\Xi_\top$), ($\mathsf{F}_{\mathscr{T}}$) and coherence. Hence $B \rightsquigarrow_{\mathscr{T}} \tau_2$ for some $\tau_2 \in \mathscr{T}$.

We show that there is $f \in \mathcal{D}_{\tau_1 \to \tau_2}$ with $X \rightsquigarrow_{\tau_1 \to \tau_2} f$ and $Y \rightsquigarrow_{\tau_1 \to \tau_2} f$. Let $d \in \mathcal{D}_{\tau_1}$. Then $Ad \rightsquigarrow_o \top$ by Lemma 6.2.12, because $A \rightsquigarrow_{\mathscr{T}} \tau_1$. So $\mathsf{Q}_L B(Xd)(Yd) \rightsquigarrow_o \top$ and by Lemma 6.2.17

there is $b_d \in \mathcal{D}_{\tau_2}$ with $Xd \rightsquigarrow_{\tau_2} b_d$ and $Yd \rightsquigarrow_{\tau_2} b_d$. Thus by $(\mathcal{F}_{\tau_1 \to \tau_2})$ we may take $f \in \mathcal{D}_{\tau_1 \to \tau_2}$ with $f^{\mathcal{N}}(d) \equiv b_d$ for $d \in \mathcal{D}_{\tau_1}$.

By Lemma 6.2.14 we obtain $\mathsf{F}AB \rightsquigarrow_{\mathscr{T}} \tau_1 \to \tau_2$, so $\mathsf{F}(\mathsf{F}AB)\mathsf{H} \rightsquigarrow_{\mathscr{T}} (\tau_1 \to \tau_2) \to o$. Let $p \in \mathcal{D}_{(\tau_1 \to \tau_2) \to o}$. We have $pf \rightsquigarrow_o \top$ or $pf \rightsquigarrow_o \bot$, by Definition 6.2.1. Thus $\neg(pf) \vee pf \rightsquigarrow_o \top$. By invariance $\neg(pX) \vee pY \rightsquigarrow_o \top$. Since $p \in \mathcal{D}_{(\tau_1 \to \tau_2) \to o}$ was arbitrary, $\mathsf{Q}_L(\mathsf{F}AB)XY \rightsquigarrow_o \top$ by $(\Xi_\top)$ and coherence. □

Lemma 6.2.19. *If $X, Y \rightsquigarrow_o \top$ or $X, Y \rightsquigarrow_o \bot$ then $\mathsf{Q}_L\mathsf{H}XY \rightsquigarrow_o \top$.*

Proof. For concreteness, assume $X \rightsquigarrow_o \top$ and $Y \rightsquigarrow_o \top$. Recall that

$$\mathsf{Q}_L\mathsf{H}XY =_\beta \Xi(\mathsf{FHH})(\lambda p.\neg(pX) \vee pY).$$

By Lemma 6.2.14 we have $\mathsf{FHH} \rightsquigarrow_{\mathscr{T}} o \to o$. Let $p \in \mathcal{D}_{o \to o}$. It suffices to show that $pX \rightsquigarrow_o \bot$ or $pY \rightsquigarrow_o \top$. If $p^{\mathcal{N}}(\top) = \top$ then $p\top \to_\gamma \top$, and thus $pY \rightsquigarrow_o \top$ by invariance. If $p^{\mathcal{N}}(\top) = \bot$ then $p\top \to_\gamma \bot$, and thus $pX \rightsquigarrow_o \bot$ by invariance. □

Definition 6.2.20. Define $\mathcal{M}_{\mathcal{N}} = \langle \mathcal{C}, I, \mathcal{T}, \mathcal{F} \rangle$ where:

- $\mathcal{C}$ is the extensional higher-order illative combinatory algebra constructed from the $\beta\eta\gamma$-equality equivalence classes of terms, with $\mathsf{k} = [\mathsf{K}]$, $\mathsf{s} = [\mathsf{S}]$, $\Xi = [\Xi]$, etc., where by $[X]$ we denote the equivalence class of X,
- I is defined by $I(c) = [c]$ for $c \in \Sigma$,
- $\mathcal{T} = \{[X] \mid X \rightsquigarrow_o \top\}$,
- $\mathcal{F} = \{[X] \mid X \rightsquigarrow_o \bot\}$.

Theorem 6.2.21. *The structure $\mathcal{M}_{\mathcal{N}}$ from Definition 6.2.20 is an $e\mathcal{I}\mathrm{K}\omega$-model such that for every $d \in \mathcal{D}_\tau$ there exists $\overline{d} \in \mathcal{C}$ so that:*

- $\overline{f} \cdot \overline{d} = \overline{f(d)}$ *for $f \in \mathcal{D}_{\tau_1 \to \tau_2}$, $d \in \mathcal{D}_{\tau_1}$,*
- $I_{\mathcal{M}}(c) = \overline{I_{\mathcal{N}}(c)}$ *for c a constant in the language of higher-order logic.*

Proof. Using Lemma 6.2.6, Lemma 6.2.13, Lemma 6.2.15, Lemma 6.2.18 and Lemma 6.2.19 it is straightforward to check the conditions for an $e\mathcal{I}\mathrm{K}\omega$-model from Definition 6.1.5. The additional conditions in the statement of the theorem follow from definitions. □

Corollary 6.2.22. *The system $e\mathcal{I}\mathrm{K}\omega$ is consistent, i.e., $\nvdash_{e\mathcal{I}\mathrm{K}\omega} \bot$.*

6.3 Translations

In this section we give sound translations of systems of higher-order logic into corresponding illative systems. The translations are extensions of those from Section 5.3. We show soundness syntactically. We also derive a limited completeness result with respect to standard semantics for higher-order logic.

In what follows $X, Y, Z, \ldots$ stand for terms from $\mathbb{T}$, and $t, s, \ldots$ stand for terms of higher-order logic (NKω), and $\varphi, \psi, \ldots$ stand for higher-order formulas (terms of type o), and $\Delta, \Delta', \ldots$ stand for sets of formulas. We assume that all constants from the syntax of traditional higher-order logic occur as constants in $\mathbb{T}$, and also all variables of traditional systems occur as variables in $\mathbb{T}$. Sometimes we write, e.g., Δ, φ instead of $\Delta \cup \{\varphi\}$.

Like in the previous section, we assume that the set of base types $\mathcal{B}$ of $e\mathcal{I}$Kω ($\mathcal{I}$Kω) is the same as the set of base types for traditional higher-order logic. For each base type $\tau \in \mathcal{B}$ there is a constant A_τ in $\mathbb{T}$. For other types $\tau \in \mathscr{T}$ we define A_τ by induction on the structure of τ:

- $\mathsf{A}_o \equiv \mathsf{H}$,
- $\mathsf{A}_{\tau_1 \to \tau_2} \equiv \mathsf{F}\mathsf{A}_{\tau_1}\mathsf{A}_{\tau_2}$.

Definition 6.3.1. We define a mapping $\lceil - \rceil$ from higher-order terms and formulas to the set of illative terms $\mathbb{T}$, and a context-providing mapping $\Gamma(-)$ from sets of higher-order terms and formulas to sets of terms from $\mathbb{T}$. The definition of $\lceil - \rceil$ is by induction on the structure of its argument:

- $\lceil x \rceil \equiv x$, for x a variable,
- $\lceil c \rceil \equiv c$, for c a constant,
- $\lceil t_1 t_2 \rceil \equiv \lceil t_1 \rceil \lceil t_2 \rceil$,
- $\lceil \lambda x.t \rceil \equiv \lambda x.\lceil t \rceil$,
- $\lceil \varphi \to \psi \rceil \equiv \lceil \varphi \rceil \supset \lceil \psi \rceil$,
- $\lceil \forall x : \tau \,.\, \varphi \rceil \equiv \Xi \mathsf{A}_\tau \lambda x.\lceil \varphi \rceil$ if $x \in V_\tau$.

We extend the mapping $\lceil - \rceil$ to sets of higher-order formulas thus: $\lceil \Delta \rceil = \{\lceil \varphi \rceil \mid \varphi \in \Delta\}$.

For a set of higher-order terms and formulas Δ, the set $\Gamma(\Delta)$ is defined to contain:

- $\mathsf{A}_\tau c$ for each $c \in \Sigma_\tau$,
- $\mathsf{A}_\tau x$ for each $x \in \mathrm{FV}(\Delta)$ with $x \in V_\tau$,
- $\mathsf{A}_\tau y$ for each $\tau \in \mathcal{B}$ and a fresh variable y.

The last point is necessary, because in ordinary higher-order logic each base type is assumed to be non-empty. If t is a term of higher-order logic, we write $\Gamma(t)$ for $\Gamma(\{t\})$.

Lemma 6.3.2. $\lceil t \rceil [x/\lceil s \rceil] \equiv \lceil t[x/s] \rceil$.

Proof. Induction on the structure of t. □

Lemma 6.3.3. $\vdash_{\mathcal{I}\mathrm{K}\omega} \mathsf{L}\mathsf{A}_\tau$ *for* $\tau \in \mathscr{T}$.

Proof. Induction on τ. □

Lemma 6.3.4. *If* $t \in T_\tau$ *then* $\Gamma(t) \vdash_{\mathcal{I}\mathrm{K}\omega} \mathsf{A}_\tau \lceil t \rceil$.

Proof. Induction on the structure of t. If $t \equiv c$ then $\lceil c \rceil \equiv c$ and $\mathsf{A}_\tau c \in \Gamma(t)$. If $t \equiv x$ then $\lceil x \rceil \equiv x$ and $\mathsf{A}_\tau x \in \Gamma(t)$.

If $t \equiv t_1 t_2$ then $t_1 \in T_{\tau_1 \to \tau_2}$ and $t_2 \in T_{\tau_1}$ for some $\tau_1, \tau_2 \in \mathscr{T}$. By the IH we have $\Gamma(t_1) \vdash \mathsf{F}\mathsf{A}_{\tau_1}\mathsf{A}_{\tau_2} t_1$ and $\Gamma(t_2) \vdash \mathsf{A}_{\tau_1} t_2$. Note that $\Gamma(t_1 t_2) = \Gamma(t_1, t_2)$. Hence $\Gamma(t_1 t_2) \vdash \mathsf{A}_{\tau_2}(t_1 t_2)$.

If $t \equiv \lambda x.t_1$ with $x \in V_{\tau_1}$ and $t_1 \in T_{\tau_2}$ then $\lceil t \rceil \equiv \lambda x.\lceil t_1 \rceil$. By the inductive hypothesis $\Gamma(t_1) \vdash \mathsf{A}_{\tau_2}\lceil t_1 \rceil$. Note that $\Gamma(t_1) = \Gamma(t) \cup \{\mathsf{A}_{\tau_1} x\}$. By Lemma 6.3.3 we have $\Gamma(t) \vdash \mathsf{L}\mathsf{A}_{\tau_1}$. We may assume $x \notin \mathrm{FV}(\Gamma(t))$. Then $\Gamma(t) \vdash \mathsf{F}\mathsf{A}_{\tau_1}\mathsf{A}_{\tau_2}(\lambda x.t_1)$ by (ΞI) and (Eq). Therefore $\Gamma(t) \vdash \mathsf{A}_{\tau_1 \to \tau_2} t$.

If $t \equiv \varphi \to \psi$ then $\lceil t \rceil \equiv \lceil \varphi \rceil \supset \lceil \psi \rceil$. By the IH we have $\Gamma(\varphi) \vdash \mathsf{H}\lceil \varphi \rceil$ and $\Gamma(\psi) \vdash \mathsf{H}\lceil \psi \rceil$. Since $\Gamma(\varphi, \psi) = \Gamma(\varphi \to \psi)$, this implies $\Gamma(\varphi \to \psi) \vdash \mathsf{H}\lceil \varphi \to \psi \rceil$.

If $t \equiv \forall x : \tau \,.\, \varphi$ then $\lceil t \rceil \equiv \Xi\mathsf{A}_\tau \lambda x.\lceil \varphi \rceil$. By the inductive hypothesis we have $\Gamma(\varphi) \vdash \mathsf{H}\lceil \varphi \rceil$. Since $\Gamma(\varphi) = \Gamma(\forall x : \tau \,.\, \varphi) \cup \{\mathsf{A}_\tau x\}$, and $\Gamma(\forall x : \tau \,.\, \varphi) \vdash \mathsf{LH}$ and we may assume that $x \notin \mathrm{FV}(\Gamma(\forall x : \tau \,.\, \varphi))$, we have $\Gamma(\forall x : \tau \,.\, \varphi) \vdash \mathsf{H}(\Xi\mathsf{A}_\tau \lambda x.\lceil \varphi \rceil)$ by (ΞHI) and (Eq). □

Lemma 6.3.5. *If $\tau \in \mathscr{T}$ then there exists a term $X \in \mathbb{T}$ with $\Gamma(\emptyset) \vdash_{\mathcal{I}\mathrm{K}\omega} \mathsf{A}_\tau X$.*

Proof. Induction on τ. If $\tau \in \mathcal{B}$ then $\mathsf{A}_\tau y \in \Gamma(\emptyset)$ for some variable y and we may take $X \equiv y$. If $\tau = o$ then we may take $X \equiv \bot$. Otherwise $\tau = \tau_1 \to \tau_2$ and by the IH there is Y with $\Gamma(\emptyset) \vdash \mathsf{A}_{\tau_2} Y$. Then $\Gamma(\emptyset) \vdash \mathsf{F}\mathsf{A}_{\tau_1}\mathsf{A}_{\tau_2}(\mathsf{K}Y)$, so we take $X \equiv \mathsf{K}Y$. □

Theorem 6.3.6 (Soundness of the translation)**.**
If $\Delta \vdash_{\mathcal{N}} \varphi$ then $\Gamma(\Delta, \varphi), \lceil \Delta \rceil \vdash_{\mathcal{I}} \lceil \varphi \rceil$, where $\mathcal{N} = \mathrm{NK}\omega$ and $\mathcal{I} = \mathcal{I}\mathrm{K}\omega$, or $\mathcal{N} = e\mathrm{NK}\omega$ and $\mathcal{I} = e\mathcal{I}\mathrm{K}\omega$.

Proof. First, we consider the case when $\mathcal{N} = \mathrm{NK}\omega$ and $\mathcal{I} = \mathcal{I}\mathrm{K}\omega$. We proceed by induction on the length of derivation of $\Delta \vdash_{\mathrm{NK}\omega} \varphi$. We consider possible rules by which $\Delta \vdash_{\mathrm{NK}\omega} \varphi$ is derived.

(Ax) Then $\Delta = \Delta', \varphi$ and we have $\Gamma(\Delta, \varphi), \lceil \Delta' \rceil, \lceil \varphi \rceil \vdash_{\mathcal{I}\mathrm{K}\omega} \lceil \varphi \rceil$.

($\bot$E$_c$) Then $\Delta, \varphi \to \bot \vdash \bot$. By the IH we have[1] $\Gamma(\Delta, \varphi \to \bot), \lceil \Delta \rceil, \lceil \varphi \rceil \supset \lceil \bot \rceil \vdash_{\mathcal{I}\mathrm{K}\omega} \lceil \bot \rceil$. Notice that $\Gamma(\Delta, \varphi \to \bot) = \Gamma(\Delta, \varphi)$. We have $\Gamma(\varphi) \vdash \mathsf{H}\lceil \varphi \rceil$ by Lemma 6.3.4. From this one easily obtains $\Gamma(\varphi) \vdash \mathsf{H}(\lceil \varphi \rceil \supset \lceil \bot \rceil)$. Hence $\Gamma(\Delta, \varphi), \lceil \Delta \rceil \vdash (\lceil \varphi \rceil \supset \lceil \bot \rceil) \supset \lceil \bot \rceil$ by (PI_l), so $\Gamma(\Delta, \varphi), \lceil \Delta \rceil \vdash \neg(\neg\lceil \varphi \rceil \vee \lceil \bot \rceil) \vee \lceil \bot \rceil$. By Lemma 4.1.19 it suffices to show

$$\Gamma(\Delta, \varphi), \lceil \Delta \rceil, \neg\lceil \varphi \rceil \vdash \bot.$$

We have

$$\Gamma(\Delta, \varphi), \lceil \Delta \rceil, \neg\lceil \varphi \rceil \vdash \neg\lceil \varphi \rceil \vee \lceil \bot \rceil.$$

Because $\lceil \bot \rceil \equiv \Xi\mathsf{HI}$, we have $\lceil \bot \rceil \vdash \bot$ by ($\bot$HI) and (ΞE). Since

$$\Gamma(\Delta, \varphi), \lceil \Delta \rceil, \neg\lceil \varphi \rceil \vdash \neg(\neg\lceil \varphi \rceil \vee \lceil \bot \rceil) \vee \lceil \bot \rceil$$

by (VE) and ($\neg$E) we obtain $\Gamma(\Delta, \varphi), \lceil \Delta \rceil, \neg\lceil \varphi \rceil \vdash \bot$.

[1] We use $\bot$ here in two distinct meanings: as a constant in the set of illative terms $\mathbb{T}$, and as the term $\bot \equiv \forall p : o \,.\, p$ in the language of NKω.

(→I) Then $\Delta, \varphi \vdash \psi$. By the IH we have $\Gamma(\Delta, \varphi, \psi), \lceil \Delta \rceil, \lceil \varphi \rceil \vdash \lceil \psi \rceil$. By Lemma 6.3.4 we have $\Gamma(\varphi) \vdash \mathsf{H}\lceil \varphi \rceil$. Since $\Gamma(\Delta, \varphi \to \psi) = \Gamma(\Delta, \varphi, \psi)$, by (PI_l) we ultimately obtain

$$\Gamma(\Delta, \varphi \to \psi), \lceil \Delta \rceil \vdash \lceil \varphi \to \psi \rceil$$

because $\lceil \varphi \to \psi \rceil \equiv \lceil \varphi \rceil \supset \lceil \psi \rceil$.

(→E) Then $\Delta \vdash \psi \to \varphi$ and $\Delta \vdash \psi$. By the IH we have $\Gamma(\Delta, \varphi, \psi), \lceil \Delta \rceil \vdash \lceil \psi \rceil \supset \lceil \varphi \rceil$ and $\Gamma(\Delta, \psi), \lceil \Delta \rceil \vdash \lceil \psi \rceil$. Hence $\Gamma(\Delta, \varphi, \psi), \lceil \Delta \rceil \vdash \lceil \varphi \rceil$. Since $\Gamma(\Delta, \varphi, \psi) = \Gamma(\Delta, \varphi) \cup \Gamma(\psi)$ we have

$$\Gamma(\Delta, \varphi), \lceil \Delta \rceil, \mathsf{A}_{\tau_1} x_1, \ldots, \mathsf{A}_{\tau_n} x_n \vdash \lceil \varphi \rceil$$

where $\{x_1, \ldots, x_n\} = \mathrm{FV}(\psi) \setminus \mathrm{FV}(\Delta, \varphi)$. By Lemma 6.3.5 there exist $X_1, \ldots, X_n$ with $\Gamma(\emptyset) \vdash \mathsf{A}_{\tau_i} X_i$. By (Sub) (see Lemma 4.1.2) we have

$$\Gamma(\Delta, \varphi), \lceil \Delta \rceil, \mathsf{A}_{\tau_1} X_1, \ldots, \mathsf{A}_{\tau_n} X_n \vdash \lceil \varphi \rceil.$$

Applying (Cut) consecutively n times we obtain

$$\Gamma(\Delta, \varphi), \lceil \Delta \rceil \vdash \lceil \varphi \rceil.$$

(∀I) Then $\varphi \equiv \forall x : \tau \,.\, \psi$ and $\Delta \vdash \psi$, where $x \notin \mathrm{FV}(\Delta)$. By the inductive hypothesis we have $\Gamma(\Delta, \psi), \lceil \Delta \rceil \vdash \lceil \psi \rceil$. Since $\Gamma(\Delta, \psi) = \Gamma(\Delta, \forall x : \tau \,.\, \psi), \mathsf{A}_\tau x$ we have

$$\Gamma(\Delta, \forall x : \tau \,.\, \psi), \lceil \Delta \rceil, \mathsf{A}_\tau x \vdash \lceil \psi \rceil.$$

Thus

$$\Gamma(\Delta, \forall x : \tau \,.\, \psi), \lceil \Delta \rceil \vdash \Xi \mathsf{A}_\tau (\lambda x. \lceil \psi \rceil).$$

(∀E) Then $\varphi \equiv \psi[x/t]$ with $t \in T_\tau$ and $\Delta \vdash \forall x : \tau \,.\, \psi$. By the inductive hypothesis $\Gamma(\Delta, \forall x : \tau \,.\, \psi), \lceil \Delta \rceil \vdash \Xi \mathsf{A}_\tau \lambda x. \lceil \psi \rceil$. By Lemma 6.3.4 we have $\Gamma(t) \vdash \mathsf{A}_\tau \lceil t \rceil$. Thus

$$\Gamma(\Delta, \forall x : \tau \,.\, \psi), \lceil \Delta \rceil \vdash \lceil \psi \rceil [x / \lceil t \rceil]$$

by (ΞE). So by Lemma 6.3.2 we have

$$\Gamma(\Delta, \forall x : \tau \,.\, \psi), \lceil \Delta \rceil \vdash \lceil \psi[x/t] \rceil.$$

Since $\Gamma(\Delta, \forall x : \tau \,.\, \psi) \subseteq \Gamma(\Delta, \psi[x/t])$, we finally obtain

$$\Gamma(\Delta, \psi[x/t]), \lceil \Delta \rceil \vdash \lceil \psi[x/t] \rceil.$$

(conv) Follows from rule (Eq).

To show the case when $\mathcal{N} = e\mathrm{NK}\omega$ and $\mathcal{I} = e\mathcal{I}\mathrm{K}\omega$ it now suffices to prove that the translations of the axioms e_f and e_b (see Definition 2.4.11) are derivable in $e\mathcal{I}\mathrm{K}\omega$. This is straightforward using (Ext_f) and (Ext_b). □

Completeness of the above translation is an open problem, i.e., we do not know whether $\Gamma(\Delta,\varphi), \lceil\Delta\rceil \vdash_{\mathcal{I}} \lceil\varphi\rceil$ implies $\Delta \vdash_{\mathcal{N}} \varphi$, where $\mathcal{N} = \mathrm{NK}\omega$ and $\mathcal{I} = \mathcal{I}\mathrm{K}\omega$, or $\mathcal{N} = e\mathrm{NK}\omega$ and $\mathcal{I} = e\mathcal{I}\mathrm{K}\omega$. However, we have the following partial result.

Theorem 6.3.7 (Completeness of the translation with respect to standard semantics)**.** *If* $\Gamma(\Delta,\varphi), \lceil\Delta\rceil \models_{e\mathcal{I}\mathrm{K}\omega} \lceil\varphi\rceil$ *then* $\Delta \models_{\mathrm{std}} \varphi$.

Proof. Let $\mathcal{N} = \langle\{\mathcal{D}_\tau \mid \tau \in \mathscr{T}\}, I_{\mathcal{N}}\rangle$ be a standard model for higher-order logic and ρ an $\mathcal{N}$-valuation such that $\mathcal{N},\rho \models_{\mathrm{std}} \Delta$ but $\mathcal{N},\rho \not\models_{\mathrm{std}} \varphi$. Let $\mathcal{M} = \langle\mathcal{C}, \underline{I_{\mathcal{M}}, \mathcal{T}}, \mathcal{F}\rangle$ be the model from Theorem 6.2.21. Let $\overline{\rho}$ be an $\mathcal{M}$-valuation defined by: $\overline{\rho}(x) = \underline{\rho(x)}$.

For a term t in the syntax of traditional higher-order logic, by $\lceil t\rceil^\rho$ we denote $\lceil t\rceil$ with each free variable x replaced by $\rho(x)$ treated as a constant in the set of terms $\mathbb{T}$ from which the model is built.

First, we show by induction on the structure of $t \in T_\tau$ that

$(\star)$ $\lceil t\rceil^\rho \rightsquigarrow_\tau [\![t]\!]^{\mathcal{N}}_\rho$,

where $\rightsquigarrow_\tau$ is as in Definition 6.2.20. If $t \equiv c$ then $c \succ_\tau c$ by Lemma 6.2.11. If $t \equiv x$ then $\rho(x) \succ_\tau \rho(x)$ by $(\mathcal{D}_\tau)$.

If $t \equiv t_1 t_2$ with $t_1 \in T_{\tau_1\to\tau_2}$ and $t_2 \in T_{\tau_1}$ then $\lceil t_1\rceil^\rho \rightsquigarrow_{\tau_1\to\tau_2} [\![t_1]\!]^{\mathcal{N}}_\rho$ and $\lceil t_2\rceil^\rho \rightsquigarrow_{\tau_1} [\![t_2]\!]^{\mathcal{N}}_\rho$, by the inductive hypothesis. By (γ) and $(\mathcal{F}_{\tau_1\to\tau_2})$, we have $\lceil t_1 t_2\rceil^\rho \rightsquigarrow_{\tau_2} [\![t_1 t_2]\!]^{\mathcal{N}}_\rho$.

If $t \equiv \lambda x.t'$ with $x \in V_{\tau_1}$ and $t' \in T_{\tau_2}$, then by the IH $\lceil t'\rceil^{\rho[x/d]} \rightsquigarrow_{\tau_2} [\![t']\!]^{\mathcal{N}}_{\rho[x/d]}$ for every $d \in \mathcal{D}_{\tau_1}$. Hence $(\lambda x.\lceil t'\rceil^\rho)d \rightsquigarrow_{\tau_2} ([\![\lambda x.t']\!]^{\mathcal{N}}_\rho)^{\mathcal{N}}(d)$ for every $d \in \mathcal{D}_{\tau_1}$. Therefore by $(\mathcal{F}_{\tau_1\to\tau_2})$ we obtain $\lceil\lambda x.t'\rceil^\rho \equiv \lambda x.\lceil t'\rceil^\rho \rightsquigarrow_{\tau_1\to\tau_2} [\![\lambda x.t']\!]^{\mathcal{N}}_\rho$.

If $t \equiv \forall x : \tau\,.\,\varphi$ then $\lceil\lambda x.\varphi\rceil^\rho d \rightsquigarrow_o [\![\varphi]\!]^{\mathcal{N}}_{\rho[x/d]} \in \{\top,\bot\}$ for every $d \in \mathcal{D}_\tau$. Therefore $\lceil\forall x : \tau\,.\,\varphi\rceil^\rho \rightsquigarrow_o [\![\forall x : \tau\,.\,\varphi]\!]^{\mathcal{N}}_\rho$ by $(\Xi_\top)$ or $(\Xi_\bot)$.

If $t \equiv \varphi \supset \psi$ then the claim follows from the inductive hypothesis, $(\neg_\top)$, $(\neg_\bot)$, $(\mathsf{V}_\top)$ and $(\mathsf{V}_\bot)$.

This concludes the proof of $(\star)$.

Now if $\mathbb{T} = \mathbb{T}_{\mathrm{CL}}$ then for $t \in T_\tau$ one easily shows by induction on the structure of $\lceil t\rceil$ that $[\![\lceil t\rceil]\!]^{\mathcal{M}}_{\overline{\rho}} = [\lceil t\rceil^\rho]$, where $[X]$ denotes the $\beta\eta\gamma$-equivalence class as in Definition 6.2.20. If $\mathbb{T} \neq \mathbb{T}_{\mathrm{CL}}$ then for $t \in T_\tau$ we have $[\![\lceil t\rceil]\!]^{\mathcal{M}}_{\overline{\rho}} = [\![(\lceil t\rceil)_{\mathrm{CL}}]\!]^{\mathcal{M}}_{\overline{\rho}} = [((\lceil t\rceil^\rho)_{\mathrm{CL}})_\lambda] = [\lceil t\rceil^\rho]$.

Since $\mathcal{T} = \{[X] \mid X \rightsquigarrow_o \top\}$, $\mathcal{F} = \{[X] \mid X \rightsquigarrow_o \bot\}$ (see Definition 6.2.20), the condition $(\star)$ and $\mathcal{T} \cap \mathcal{F} = \emptyset$ imply:

- $[\![\lceil t\rceil]\!]^{\mathcal{M}}_{\overline{\rho}} \in \mathcal{T}$ iff $[\![t]\!]^{\mathcal{N}}_\rho = \top$,
- $[\![\lceil t\rceil]\!]^{\mathcal{M}}_{\overline{\rho}} \in \mathcal{F}$ iff $[\![t]\!]^{\mathcal{N}}_\rho = \bot$.

From this it follows that $\mathcal{M},\overline{\rho} \models \Gamma(\Delta,\varphi), \lceil\Delta\rceil$ but $\mathcal{M},\overline{\rho} \not\models \lceil\varphi\rceil$. □

Corollary 6.3.8. *If* $\Gamma(\Delta,\varphi), \lceil\Delta\rceil \vdash_{e\mathcal{I}\mathrm{K}\omega} \lceil\varphi\rceil$ *then* $\Delta \models_{\mathrm{std}} \varphi$.

Chapter 7

Extensions

In this chapter we introduce the system $\mathcal{I}^+$ which is an extension of the system $e\mathcal{I}\mathrm{K}\omega_{\mathrm{CL}w}$ from the previous chapter by a choice operator, universal and empty types, the conditional combinator, subtypes, dependent function types, dependent sums and W-types. The system $\mathcal{I}^+$ may interpret a great deal of mathematics. We study only a version of $\mathcal{I}^+$ based on combinatory logic with weak equality, in order to avoid some complications in the model construction. The incorporation of β- and η-reduction adds some tedious technicalities which obscure the main ideas of the construction.

7.1 Illative system

Definition 7.1.1. The set of terms $\mathbb{T}$ of the system $\mathcal{I}^+$ is defined as $\mathbb{T}_{\mathrm{CL}}(\Sigma)$ where Σ contains the following illative constants: Ξ, $\mathsf{\Lambda}$, V, $\neg$, $\bot$, ϵ, M, W, sup, T, D. We adopt the abbreviations from Definition 6.1.1, except the one for F, plus the following:

- $\mathsf{G} \equiv \lambda xyf.\Xi x(\lambda z.yz(fz))$,
- $\mathsf{F} \equiv \lambda xy.\mathsf{G}x(\mathsf{K}y)$,
- $\pi \equiv \lambda xyz.zxy$,
- $\pi_1 \equiv \lambda x.x\mathsf{K}$,
- $\pi_2 \equiv \lambda x.x(\mathsf{KI})$,
- $\Upsilon \equiv \lambda xyz.xz \wedge yz$,
- $\Sigma \equiv \lambda xyz.x(\pi_1 z) \wedge y(\pi_1 z)(\pi_2 z)$,
- **if** X **then** Y **else** $Z \equiv \mathsf{M}XYZ$,
- $A \times B \equiv \Sigma A(\mathsf{K}B)$,
- $A + B \equiv \Sigma\mathsf{H}(\lambda x.$**if** x **then** A **else** $B)$,
- $\mathsf{O} \equiv \mathsf{K}\bot$,
- $\mathsf{E} \equiv \mathsf{FOO}$.

A judgement of the system $\mathcal{I}^+$ has one of two forms: $\Gamma \vdash X$ or $\Gamma \vdash X = Y$, where X, Y are terms and Γ is a finite set of terms. The rules of $\mathcal{I}^+$ are those of Figure 4.3 except (Eq), plus the rules (ΞI), (ΞE), ($\neg\Xi$I), ($\neg\Xi$E), ($\Xi\mathsf{HI}$), ($\Xi\mathsf{LE}$), (HL) from Figure 6.1 and all rules from Figure 7.1 and Figure 7.2. Recall that $X =_A Y$ is an abbreviation for $\mathsf{Q}_L AXY$ (the definition of Q_L appears in Definition 6.1.1).

Note that we use a different abbreviation for F than in Chapter 6. This is because by basing the system on combinatory logic with weak equality we have effectively disallowed reduction "under lambdas", and we need the new definition of F to make the rule (FL) admissible.

The term π represents a pair-forming operator, and π_1, π_2 are the first and second projections, respectively. Their definitions are standard. Now we shall give an informal explanation of the meaning of the new illative primitives not explained in Section 1.1 or in Chapter 6.

- M Conditional combinator. This combinator allows "branching" on arbitrary formulas. Intuitively, the term $\mathsf{M}XYZ$ should be equal to Y if X is true, or to Z if X is false. An important thing to notice is that X above need not be computable – it may represent any proposition, possibly one containing unbounded quantification. To incorporate the conditional combinator it is necessary to extend the syntax of judgments of $\mathcal{I}^+$ by judgements of the form $\Gamma \vdash X = Y$. Alternatively, instead of introducing a new form of judgement $\Gamma \vdash X = Y$ we could introduce a new combinator for equality inside the system. This approach was adopted in [Cza13c, Cza13d]. However, such a choice complicates the model construction.

- ϵ Choice operator. Intuitively, ϵAX is an object of type A satisfying X, if such an object exists, or an arbitrary object of type A otherwise. If A is empty then ϵAX is undefined.

- O Empty type. Using the empty type O and the functionality combinator F one may define the universal type E by $\mathsf{E} \equiv \mathsf{FOO}$. Indeed, every object X is a function from O to O, because for every object Y of type O (and there none) the object XY is of type O.

- Υ Subtype constructor. A term ΥAX is interpreted as the subtype of A consisting of all objects Y of type A such that XY is true.

- Σ Dependent sum type constructor. A term ΣAB represents a dependent sum type – the type of all pairs πXY such that X has type A and Y has type BX. Using dependent sums one may define binary products $A \times B$ and non-dependent binary sums $A + B$.

- W W-type constructor. A term $\mathsf{W}AB$ is interpreted as a W-type: the type of all well-founded trees with nodes labelled with objects of the *constructor type* A and branching specified by the *selector family* B, i.e., a node labelled with a has a distinct child for each object of type Ba. If we have an object a of type A, i.e. a particular label, and if we have a function b from Ba to $\mathsf{W}AB$, i.e. a collection of subtrees, then we may form the tree $\sup(\mathsf{W}AB)ab$.

  W-types originally appeared in Martin-Löf's type theory [ML84],[NPS90, Chapter 15]. Using W-types it is possible to define many inductive types. For example, to define the type of natural numbers, we need a type A with exactly two elements z and s, and a family B such that Bz is empty and Bs has exactly one element. Then $\mathsf{W}AB$ represents the type of natural numbers. The tree corresponding to the natural number n consists of n nodes labelled by s (the first of these is the root), one after another, ending in one more node (a leaf) labelled by z (if the number is zero, then the root is the leaf).

T Test combinator for W-types. This combinator allows to test the labels of nodes in a tree which is an element of a W-type. Intuitively, if $X =_A X'$ is true (X is equal to X' in type A, i.e., $\mathsf{Q}_L AXX'$ holds, see Definition 6.1.1) then $\mathsf{T}(\mathrm{sup}(\mathsf{W}AB)XY)X'$ is true (provided $(\mathrm{sup}(\mathsf{W}AB)XY)$ has the type $\mathsf{W}AB$). If $X =_A X'$ is false, then $\mathsf{T}(\mathrm{sup}(\mathsf{W}AB)XY)X'$ is false.

D Destructor combinator for W-types. This combinator allows to destruct a tree which is a member of a W-type, i.e., to obtain its subtrees. Intuitively, if $\mathrm{sup}(\mathsf{W}AB)XY$ has type $\mathsf{W}AB$ and Z has type BX, then $\mathsf{D}(\mathrm{sup}(\mathsf{W}AB)XY)Z$ is identical with YZ, or in other words $\mathsf{D}(\mathrm{sup}(\mathsf{W}AB)XY)Z$ is the child of $\mathrm{sup}(\mathsf{W}AB)XY$ associated with the object Z.

One may wonder why we chose to include the above illative primitives and not some others, e.g. an illative primitive representing a constructor of a power type (the type of all subtypes of a given type). The answer is that (most of) the listed primitives correspond to types known from type theory, they make sense in a constructive setting (subtypes, dependent types and W-types essentially appear in Martin-Löf's type theory), and they suffice to interpret a great deal of mathematics. Actually, it seems highly plausible that the model construction in Section 7.2 could be adapted for an illative system incorporating virtually any notion from standard set theory. We leave for future work the problem of incorporating in a conceptually satisfactory way the notions of set theory into an illative system.

Most of the rules from Figure 7.1 are self-explanatory. They implement the intuitions about the illative primitives explained above. Note that the induction rule (WInd) for W-types in Figure 7.1 is unrestricted, i.e., the term X is not required a priori to have any particular type. One can thus, e.g., reason about types of terms by induction.

Lemma 7.1.2. *The rules from Figure 4.1, rule* (EM) *from Definition 4.1.1, and rules* (XI)*,* (XE)*,* (XHI) *and* (XLE) *form Figure 5.1, are all admissible in* $\mathcal{I}^+$*.*

Proof. Follows directly from Lemma 6.1.2. ☐

Lemma 7.1.3. *The rule* (FL) *from Figure 6.1 and all rules from Figure 7.3 are admissible in* $\mathcal{I}^+$*.*

Proof. Follows directly from definitions. ☐

Therefore, all rules of $e\mathcal{I}\mathrm{K}\omega$ except $(\mathsf{A}_\tau\mathsf{L})$ are admissible in $\mathcal{I}^+$. The system $\mathcal{I}^+$ is thus essentially an extension of $e\mathcal{I}\mathrm{K}\omega$, but without the base types $\mathcal{B}$.

Lemma 7.1.4. *If* $\Gamma \vdash X =_A Y$*,* $\Gamma \vdash \mathsf{F}ABZ$*,* $\Gamma \vdash AX$ *and* $\Gamma \vdash \mathsf{L}B$*, then* $\Gamma \vdash ZX =_B ZY$*.*

$$\frac{\Gamma \vdash \mathsf{L}A \quad \Gamma, Ax \vdash \mathsf{L}(Bx) \quad x \notin \mathrm{FV}(\Gamma, A, B)}{\Gamma \vdash \mathsf{L}(\mathsf{G}AB)}\ (\mathsf{GL})$$

$$\frac{}{\Gamma \vdash \mathsf{LO}}\ (\mathsf{OL}) \qquad \frac{\Gamma \vdash \mathsf{L}A \quad \Gamma \vdash \mathsf{F}A\mathsf{L}B}{\Gamma \vdash \mathsf{L}(\Sigma AB)}\ (\Sigma\mathsf{L})$$

$$\frac{\Gamma \vdash \mathsf{L}A \quad \Gamma \vdash \mathsf{F}A\mathsf{L}B}{\Gamma \vdash \mathsf{L}(\mathsf{W}AB)}\ (\mathsf{WL}) \qquad \frac{\Gamma \vdash \mathsf{L}A \quad \Gamma \vdash \mathsf{F}A\mathsf{H}X}{\Gamma \vdash \mathsf{L}(\Upsilon AX)}\ (\Upsilon\mathsf{L})$$

$$\frac{\Gamma \vdash X \quad \Gamma \vdash X = Y}{\Gamma \vdash Y}\ (\mathrm{Eq})$$

$$\frac{X =_w Y}{\Gamma \vdash X = Y}\ (\mathrm{EqI}) \qquad \frac{\Gamma \vdash Y = Z}{\Gamma \vdash XY = XZ}\ (\mathrm{EqC})$$

$$\frac{\Gamma \vdash X = Y}{\Gamma \vdash Y = X}\ (\mathrm{EqS}) \qquad \frac{\Gamma \vdash X = Y \quad \Gamma \vdash Y = Z}{\Gamma \vdash X = Z}\ (\mathrm{EqT})$$

$$\frac{\Gamma \vdash X}{\Gamma \vdash \mathsf{M}X = \mathsf{K}}\ (\mathsf{MI}_1) \qquad \frac{\Gamma \vdash \neg X}{\Gamma \vdash \mathsf{M}X = \mathsf{KI}}\ (\mathsf{MI}_2)$$

$$\frac{\Gamma \vdash \mathsf{X}AX \quad \Gamma \vdash \mathsf{F}A\mathsf{H}X}{\Gamma \vdash X(\epsilon AX)}\ (\epsilon \mathrm{I}_l) \qquad \frac{\Gamma \vdash \mathsf{X}AA \quad \Gamma \vdash \mathsf{F}A\mathsf{H}X}{\Gamma \vdash A(\epsilon AX)}\ (\epsilon \mathrm{I}_r)$$

$$\frac{\Gamma \vdash AX \quad \Gamma \vdash \mathsf{F}(BX)(\mathsf{W}AB)Y \quad \Gamma \vdash \mathsf{L}(\mathsf{W}AB)}{\Gamma \vdash \mathsf{W}AB(\sup(\mathsf{W}AB)XY)}\ (\mathsf{WI})$$

$$\frac{\Gamma \vdash \mathsf{W}AB(\sup(\mathsf{W}AB)XY) \quad \Gamma \vdash X =_A Z}{\Gamma \vdash \mathsf{T}(\sup(\mathsf{W}AB)XY)Z}\ (\mathsf{TI}_1)$$

$$\frac{\Gamma \vdash \mathsf{W}AB(\sup(\mathsf{W}AB)XY) \quad \Gamma \vdash \neg(X =_A Z)}{\Gamma \vdash \neg(\mathsf{T}(\sup(\mathsf{W}AB)XY)Z)}\ (\mathsf{TI}_2)$$

$$\frac{\Gamma \vdash \mathsf{W}AB(\sup(\mathsf{W}AB)XY) \quad \Gamma \vdash BXZ}{\Gamma \vdash \mathsf{D}(\sup(\mathsf{W}AB)XY)Z = YZ}\ (\mathsf{DI})$$

$$\frac{\Gamma \vdash \mathsf{L}(\mathsf{W}AB) \quad \Gamma, Ax, \mathsf{F}(Bx)(\mathsf{W}AB)y, \forall z{:}Bx.X(yz) \vdash X(\sup(\mathsf{W}AB)xy)}{\Gamma \vdash \Xi(\mathsf{W}AB)X}\ (\mathsf{WInd})$$

(in (WInd) we assume $x, y, z \notin \mathrm{FV}(\Gamma, A, B, X)$)

Figure 7.1: Additional rules for $\mathcal{I}^+$

$$\frac{\Gamma \vdash \forall x : A \,.\, Xx =_{Bx} Yx \quad x \notin \mathrm{FV}(X, Y, A, B)}{\Gamma \vdash X =_{\mathsf{G}AB} Y} \ (\mathrm{Ext}_f)$$

$$\frac{\Gamma \vdash X \supset Y \quad \Gamma \vdash Y \supset X}{\Gamma \vdash X =_{\mathsf{H}} Y} \ (\mathrm{Ext}_b)$$

$$\frac{\Gamma \vdash \pi_1 X =_A \pi_1 Y \quad \Gamma \vdash \pi_2 X =_{B(\pi_1 X)} \pi_2 Y \quad \Gamma \vdash \mathsf{L}(\Sigma AB)}{\Gamma \vdash X =_{\Sigma AB} Y} \ (\mathrm{Ext}_s)$$

$$\frac{\Gamma \vdash X =_A X' \quad \Gamma \vdash Y =_{\mathsf{F}(BX)(\mathsf{W}AB)} Y' \quad \Gamma \vdash \mathsf{L}(\mathsf{W}AB)}{\Gamma \vdash \sup(\mathsf{W}AB)XY = \sup(\mathsf{W}AB)X'Y'} \ (\mathrm{Ext}_w)$$

Figure 7.2: Extensionality rules for $\mathcal{I}^+$

Proof. Identical to the proof of Lemma 6.1.3. □

The following proposition shows that, because of the presence of the conditional combinator M, all functions whose domain is the universal type E are essentially constant functions. This property of the system $\mathcal{I}^+$ may seem paradoxical at first sight, but it only means that quantifying over the whole universe does not make much sense. It should be kept in mind that, intuitively, the universe contains some nonsensical, meaningless objects, like for instance an X such that $X = \neg X$. Usually, one just considers objects which have some "reasonable" types.

Proposition 7.1.5. *If $\Gamma \vdash \mathsf{L}A$ and for every X we have $\Gamma \vdash A(FX)$, then for all X, Y we have $\Gamma \vdash FX =_A FY$.*

Proof. Let X, Y be arbitrary terms. Define Z by the following recursive equation:

$$Z = \textbf{if } FX =_A FZ \textbf{ then } Y \textbf{ else } X$$

Because $\Gamma \vdash A(FZ)$ and $\Gamma \vdash A(FX)$, we have $\Gamma, \mathsf{F}A\mathsf{H}p \vdash \mathsf{H}(p(FX) \supset p(FZ))$ where $p \notin \mathrm{FV}(\Gamma, A, F, X, Z)$. Since $\Gamma \vdash \mathsf{L}A$ and $\Gamma \vdash \mathsf{L}\mathsf{H}$, we have $\Gamma \vdash \mathsf{L}(\mathsf{F}A\mathsf{H})$, and thus

$$\Gamma \vdash \mathsf{H}(\forall p : \mathsf{F}A\mathsf{H} \,.\, p(FX) \supset p(FZ))$$

by (ΞHI) and the rules for equality. Thus $\Gamma \vdash \mathsf{H}(FX =_A FZ)$. Of course

$$\Gamma, FX =_A FZ \vdash FX =_A FZ.$$

Also $\Gamma, FX =_A FZ \vdash Z = Y$ by (MI_1) and the rules for equality. Hence

$$\Gamma, FX =_A FZ \vdash FX =_A FY.$$

We also have $\Gamma, \neg(FX =_A FZ) \vdash Z = X$ by (MI_2) and the rules for equality. Because $\Gamma \vdash A(FX)$ and $\Gamma \vdash \mathsf{L}(\mathsf{F}A\mathsf{H})$, we have $\Gamma \vdash FX =_A FX$ using (ΞI), the rules for propositional

$$\frac{\Gamma \vdash \mathsf{O}X}{\Gamma \vdash Y}\ (\mathsf{O}\mathrm{E}) \qquad \frac{}{\Gamma \vdash \mathsf{E}X}\ (\mathsf{EI}) \qquad \frac{}{\Gamma \vdash \mathsf{LE}}\ (\mathsf{EL})$$

$$\frac{\Gamma \vdash AY \quad \Gamma \vdash XY}{\Gamma \vdash \Upsilon AXY}\ (\Upsilon\mathrm{I}) \qquad \frac{\Gamma \vdash \Upsilon AXY}{\Gamma \vdash AY}\ (\Upsilon\mathrm{E}_1) \quad \frac{\Gamma \vdash \Upsilon AXY}{\Gamma \vdash XY}\ (\Upsilon\mathrm{E}_2)$$

$$\frac{\Gamma \vdash A(\pi_1 X) \quad \Gamma \vdash B(\pi_1 X)(\pi_2 X)}{\Gamma \vdash \Sigma ABX}\ (\Sigma\mathrm{I}) \qquad \frac{\Gamma \vdash \Sigma ABX}{\Gamma \vdash A(\pi_1 X)}\ (\Sigma\mathrm{E}_1)$$

$$\frac{\Gamma \vdash \Sigma ABX}{\Gamma \vdash B(\pi_1 X)(\pi_2 X)}\ (\Sigma\mathrm{E}_2)$$

$$\frac{\Gamma \vdash A(\pi_1 X) \quad \Gamma \vdash B(\pi_2 X)}{\Gamma \vdash (A \times B)X}\ (\times\mathrm{I}) \qquad \frac{\Gamma \vdash (A \times B)X}{\Gamma \vdash A(\pi_1 X)}\ (\times\mathrm{E}_1)$$

$$\frac{\Gamma \vdash (A \times B)X}{\Gamma \vdash B(\pi_2 X)}\ (\times\mathrm{E}_2)$$

$$\frac{\Gamma \vdash \mathsf{L}A \quad \Gamma \vdash \mathsf{L}B}{\Gamma \vdash \mathsf{L}(A \times B)}\ (\times\mathsf{L})$$

$$\frac{\Gamma \vdash AX \quad \Gamma \vdash Z}{\Gamma \vdash (A+B)(\pi Z X)}\ (+\mathrm{I}_1) \qquad \frac{\Gamma \vdash BX \quad \Gamma \vdash \neg Z}{\Gamma \vdash (A+B)(\pi Z X)}\ (+\mathrm{I}_2)$$

$$\frac{\Gamma \vdash (A+B)X \quad \Gamma, \pi_1 X, A(\pi_2 X) \vdash Y \quad \Gamma, \neg(\pi_1 X), B(\pi_2 X) \vdash Y}{\Gamma \vdash Y}\ (+\mathrm{E})$$

$$\frac{\Gamma \vdash \mathsf{L}A \quad \Gamma \vdash \mathsf{L}B}{\Gamma \vdash \mathsf{L}(A+B)}\ (+\mathsf{L})$$

Figure 7.3: Admissible rules in $\mathcal{I}^+$

$$\frac{}{\Gamma \vdash \mathsf{N}0}\ (\mathsf{NI}_0) \qquad \frac{\Gamma \vdash \mathsf{N}X}{\Gamma \vdash \mathsf{N}(\mathsf{s}X)}\ (\mathsf{NI}_s) \qquad \frac{}{\Gamma \vdash \mathsf{LN}}\ (\mathsf{NL})$$

$$\frac{\Gamma \vdash X0 \quad \Gamma, \mathsf{N}x, Xx \vdash X(\mathsf{s}x) \quad x \notin \mathrm{FV}(\Gamma, X)}{\Gamma \vdash \Xi\mathsf{N}X}\ (\mathsf{N}\mathrm{Ind})$$

Figure 7.4: Rules for the type of natural numbers

connectives and the rules for equality. Thus $\Gamma, \neg(FX =_A FZ) \vdash FX =_A FZ$, and therefore $\Gamma, \neg(FX =_A FZ) \vdash \bot$. Hence

$$\Gamma, \neg(FX =_A FZ) \vdash FX =_A FY$$

by ($\bot$E). Therefore, since $\Gamma \vdash \mathsf{H}(FX =_A FZ)$, and $\Gamma, FX =_A FZ \vdash FX =_A FY$, and $\Gamma, \neg(FX =_A FZ) \vdash FX =_A FY$, we ultimately obtain $\Gamma \vdash FX =_A FY$ by (EM) and (VE). □

In extensional Martin-Löf's type theory, using W-types it is possible to define many inductive types [Dyb97, AAG04]. A very similar construction may be carried out in $\mathcal{I}^+$. The extensionality rules (Ext_f), (Ext_b) and (Ext_w) are essential here. Without them, when trying to derive induction principles for inductive types defined using W-types, one encounters a problem similar to the problem encountered in intensional Martin-Löf's type theory.

We will not formulate here a general theorem. We just present the example of natural numbers. The type N is defined by:

$$\mathsf{N} \equiv \mathsf{WH}(\lambda x \,.\, \textbf{if } x \textbf{ then } \Upsilon\mathsf{HI} \textbf{ else } \mathsf{O})$$

Recall from the previous informal discussion that the type of natural numbers should be represented by $\mathsf{W}AB$ where A has two elements z, s, and Bz is empty and Bs is a singleton. Because the system $\mathcal{I}^+$ is classical, the type of propositions H has two elements $\top$ and $\bot$ (up to Leibniz equality $=_{\mathsf{H}}$ in type H). The type $\Upsilon\mathsf{HI}$ is essentially a singleton – its only element is $\top$.

We use the abbreviations:

$$\begin{array}{rcl} 0 & \equiv & \sup \mathsf{N}\bot\mathsf{K} \\ \mathsf{s} & \equiv & \lambda x.\sup \mathsf{N}\top(\mathsf{K}x) \\ \mathsf{p} & \equiv & \lambda x.\mathsf{D}x\top \end{array}$$

Lemma 7.1.6. *The rules from Figure 7.4 are admissible in $\mathcal{I}^+$.*

Proof. The rule (NL) follows from (WL). Indeed, we have $\Gamma \vdash \mathsf{LH}$ by (HL) and $\Gamma \vdash \mathsf{L}(\Upsilon\mathsf{HI})$ by (ΥL) and $\Gamma \vdash \mathsf{LO}$ by (OL). Because $\mathsf{H}x \equiv x \vee \neg x$, by ($\Xi$I), ($\mathsf{V}$E), ($\mathsf{MI}_1$) and ($\mathsf{MI}_2$) we conclude $\Gamma \vdash \mathsf{FHL}B$. Therefore $\Gamma \vdash \mathsf{LN}$ by (NL).

Let $B \equiv \lambda x.\textbf{if } x \textbf{ then } \Upsilon\mathsf{HI} \textbf{ else } \mathsf{O}$. The rule ($\mathsf{NI}_0$) follows from ($\mathsf{WI}$), (Eq), ($\bot$E) and ($\mathsf{FI}$). Indeed, we have $\Gamma \vdash \mathsf{H}\bot$. Because $\Gamma \vdash B\bot = \mathsf{O}$ and $\mathsf{O} \equiv \mathsf{K}\bot$, we have $\Gamma, B\bot x \vdash \bot$ with

$x \notin \mathrm{FV}(\Gamma)$. Hence $\Gamma, B\bot x \vdash \mathsf{N}(\mathsf{K}x)$ by ($\bot$E). Thus $\Gamma \vdash \mathsf{F}(B\bot)\mathsf{NK}$ by (FI). So we conclude $\Gamma \vdash \mathsf{N}0$ using (WI), because $0 \equiv \sup \mathsf{N}\bot\mathsf{K}$ and $\Gamma \vdash \mathsf{LN}$ by (WL).

The rule (NI_s) follows by a similar argument. Indeed, assume $\Gamma \vdash \mathsf{N}X$. We have $\Gamma \vdash \mathsf{s}X = \sup \mathsf{N}\top(\mathsf{K}X)$. Of course $\Gamma \vdash \mathsf{H}\top$. By ($\mathsf{MI}_1$) and the rules for equality we have $\Gamma \vdash B\top = \Upsilon\mathsf{HI}$. Using ($\Upsilon\mathsf{L}$) we thus obtain $\Gamma \vdash \mathsf{L}(B\top)$. Because $\Gamma \vdash \mathsf{N}X$, we have $\Gamma, B\top x \vdash \mathsf{K}Xx$ where $x \notin \mathrm{FV}(\Gamma, B, X)$. Hence using ($\Xi$I) we obtain $\Gamma \vdash \mathsf{F}(B\top)\mathsf{N}(\mathsf{K}X)$. Also $\Gamma \vdash \mathsf{LN}$ by (NL). Therefore, by (WI) we conclude $\Gamma \vdash \mathsf{N}(\sup \mathsf{N}\top(\mathsf{K}X))$, i.e., $\Gamma \vdash \mathsf{N}(\mathsf{s}X)$.

We prove that (NInd) is admissible. Thus assume $\Gamma \vdash X0$ and $\Gamma, \mathsf{N}x, Xx \vdash X(\mathsf{s}x)$, where $x \notin \mathrm{FV}(\Gamma, X)$. By ($\mathsf{W}$Ind) it suffices to show that $\Gamma, \mathsf{H}x, \mathsf{F}(Bx)\mathsf{N}y, \forall z : Bx \,.\, X(yz) \vdash X(\sup \mathsf{N}xy)$ where $x, y, z \notin \mathrm{FV}(\Gamma, X)$. Since $\mathsf{H}x \equiv x \vee \neg x$, by ($\vee$E), ($\mathsf{MI}_1$), ($\mathsf{MI}_2$), (EqL) and (Weak) it suffices to show two cases:

- $\Gamma, x, \mathsf{F}(\Upsilon\mathsf{HI})\mathsf{N}y, \forall z : \Upsilon\mathsf{HI} \,.\, X(yz) \vdash X(\sup \mathsf{N}xy)$. First note that $\vdash \Upsilon\mathsf{HI}\top$. We thus have $\mathsf{F}(\Upsilon\mathsf{HI})\mathsf{N}y \vdash \mathsf{N}(y\top)$ and $\mathsf{F}(\Upsilon\mathsf{HI})\mathsf{N}y, \forall z : \Upsilon\mathsf{HI} \,.\, X(yz) \vdash X(y\top)$. Because

  $$\Gamma, \mathsf{N}(y\top), X(y\top) \vdash X(\mathsf{s}(y\top))$$

  and $\mathsf{s}(y\top) = \sup \mathsf{N}\top(\mathsf{K}(y\top))$, we have

  $$(\star) \qquad \Gamma, \mathsf{F}(\Upsilon\mathsf{HI})\mathsf{N}y, \forall z : \Upsilon\mathsf{HI} \,.\, X(yz) \vdash X(\sup \mathsf{N}\top(\mathsf{K}(y\top)))$$

  using (Sub), (Weak) and (Cut). By (Ext_b) we have $x \vdash \top =_H x$. Then also

  $$x \vdash \top =_{\Upsilon\mathsf{HI}} x,$$

  because the provability of $\mathsf{F}(\Upsilon\mathsf{HI})\mathsf{N}p$ implies the provability of $\mathsf{FHN}p$. So

  $$\Gamma, \mathsf{F}(\Upsilon\mathsf{HI})\mathsf{N}y, \Upsilon\mathsf{HI}x \vdash y\top =_{\mathsf{N}} yx$$

  by Lemma 7.1.4. Since $\mathsf{K}(y\top)x =_w y\top$, by the rules for equality we obtain

  $$\Gamma, \mathsf{F}(\Upsilon\mathsf{HI})\mathsf{N}y, \Upsilon\mathsf{HI}x \vdash \mathsf{K}(y\top)x =_{\mathsf{N}} yx.$$

  Thus by (ΞI) and (Ext_f) we have

  $$\Gamma, \mathsf{F}(\Upsilon\mathsf{HI})\mathsf{N}y \vdash \mathsf{K}(y\top) =_{\mathsf{F}(\Upsilon\mathsf{HI})\mathsf{N}} y.$$

  Since we also have $x \vdash \top =_{\mathsf{H}} x$ and $\vdash \mathsf{LN}$, by (Ext_w) we obtain

  $$\Gamma, x, \mathsf{F}(\Upsilon\mathsf{HI})\mathsf{N}y \vdash \sup \mathsf{N}\top(\mathsf{K}y\top) = \sup \mathsf{N}xy.$$

  Therefore, by ($\star$) and the rules for equality we conclude

  $$\Gamma, x, \mathsf{F}(\Upsilon\mathsf{HI})\mathsf{N}y, \forall z : \Upsilon\mathsf{HI} \,.\, X(yz) \vdash X(\sup \mathsf{N}xy)$$

- $\Gamma, \neg x, \mathsf{FON} y, \forall z : \mathsf{O} \,.\, X(yz) \vdash X(\sup \mathsf{N} xy)$. We have $\Gamma \vdash X0$, i.e., $\Gamma \vdash X(\sup \mathsf{N}\bot\mathsf{K})$. By (Ext$_b$) we have $\neg x \vdash \bot =_{\mathsf{H}} x$. We also have $\vdash \mathsf{K} =_{\mathsf{FON}} y$. Therefore

$$\neg x \vdash \sup \mathsf{N}\bot\mathsf{K} = \sup \mathsf{N} xy$$

by (Ext$_w$). So we finally conclude

$$\Gamma, \neg x, \mathsf{FON} y, \forall z : \mathsf{O} \,.\, X(yz) \vdash X(\sup \mathsf{N} xy)$$

by $\Gamma \vdash X(\sup \mathsf{N}\bot\mathsf{K})$ and the rules for equality.

□

Lemma 7.1.7. *If* $\Gamma \vdash \mathsf{N}X$ *then* $\Gamma \vdash \mathsf{p}(\mathsf{s}X) = X$.

Proof. Assume $\Gamma \vdash \mathsf{N}X$. We have $\mathsf{p}(\mathsf{s}X) =_w \mathsf{D}(\sup \mathsf{N}\top(\mathsf{K}X))\top$. Since $\Gamma \vdash \mathsf{N}(\sup \mathsf{N}\top(\mathsf{K}X))$ and $\Gamma \vdash \mathsf{K}X\top$, by (DI) we obtain $\Gamma \vdash \mathsf{D}(\sup \mathsf{N}\top(\mathsf{K}X))\top = \mathsf{K}X\top$. Using the rules for equality, we conclude $\Gamma \vdash \mathsf{p}(\mathsf{s}X) = X$. □

Definition 7.1.8. An $\mathcal{I}^+$-*model* is a tuple $\langle \mathcal{C}, I, \mathcal{T}, \mathcal{F} \rangle$ where

- $\mathcal{C}$ is a combinatory algebra.
- I is a function from the signature Σ to $\mathcal{C}$. We use the notations $\mathsf{s} = I(\mathsf{S})$, $\mathsf{k} = I(\mathsf{K})$, $\Xi = I(\Xi)$, $\epsilon = I(\epsilon)$, $\mathsf{w} = I(\mathsf{W})$, etc. We define the elements $\mathsf{g}, \mathsf{f}, \upsilon, \Sigma, \ldots \in \mathcal{C}$ in an obvious way to correspond to $\mathsf{G}, \mathsf{F}, \Upsilon, \Sigma$, etc.
- $\mathcal{T}$ and $\mathcal{F}$ are sets of elements of $\mathcal{C}$ satisfying the following for any $a, b, c, d \in \mathcal{C}$, where we use the notation $\mathcal{T}(a) = \{b \mid a \cdot b \in \mathcal{T}\}$ for $a \in \mathcal{C}$. The first 11 conditions are identical with the conditions in Definition 6.1.5.
  1. $\mathcal{T} \cap \mathcal{F} = \emptyset$,
  2. $\bot \in \mathcal{F}$,
  3. $\neg \cdot a \in \mathcal{T}$ iff $a \in \mathcal{F}$,
  4. $\neg \cdot a \in \mathcal{F}$ iff $a \in \mathcal{T}$,
  5. $\vee \cdot a \cdot b \in \mathcal{T}$ iff $a \in \mathcal{T}$ or $b \in \mathcal{T}$,
  6. $\vee \cdot a \cdot b \in \mathcal{F}$ iff $a \in \mathcal{F}$ and $b \in \mathcal{F}$,
  7. $\wedge \cdot a \cdot b \in \mathcal{T}$ iff $a \in \mathcal{T}$ and $b \in \mathcal{T}$,
  8. $\wedge \cdot a \cdot b \in \mathcal{F}$ iff $a \in \mathcal{F}$ or $b \in \mathcal{F}$,
  9. $\Xi \cdot a \cdot b \in \mathcal{T}$ iff $\mathsf{L} \cdot a \in \mathcal{T}$ and for every $c \in \mathcal{C}$ with $a \cdot c \in \mathcal{C}$ we have $b \cdot c \in \mathcal{C}$,
  10. $\Xi \cdot a \cdot b \in \mathcal{F}$ iff $\mathsf{L} \cdot a \in \mathcal{T}$ and there exists $c \in \mathcal{C}$ with $a \cdot c \in \mathcal{T}$ and $b \cdot c \in \mathcal{F}$,
  11. $\mathsf{L} \cdot \mathsf{h} \in \mathcal{T}$,
  12. $\mathsf{L} \cdot \mathsf{o} \in \mathcal{T}$,
  13. if $a \in \mathcal{T}$ then $\mathsf{M} \cdot a = \mathsf{k}$,
  14. if $a \in \mathcal{F}$ then $\mathsf{M} \cdot a = \mathsf{k} \cdot \mathsf{I}$,
  15. if $\mathsf{w} \cdot a \cdot b \cdot (\sup \cdot (\mathsf{w} \cdot a \cdot b) \cdot c \cdot d) \in \mathcal{T}$ and $\mathsf{q} \cdot a \cdot c \cdot e \in \mathcal{T}$ then $\mathsf{t} \cdot (\sup \cdot (\mathsf{w} \cdot a \cdot b) \cdot c \cdot d) \cdot e \in \mathcal{T}$,

16. if $\mathsf{w} \cdot a \cdot b \cdot (\sup \cdot (\mathsf{w} \cdot a \cdot b) \cdot c \cdot d) \in \mathcal{T}$ and $\mathsf{q} \cdot a \cdot c \cdot e \in \mathcal{F}$ then $\mathsf{t} \cdot (\sup \cdot (\mathsf{w} \cdot a \cdot b) \cdot c \cdot d) \cdot e \in \mathcal{F}$,
17. if $\mathsf{w} \cdot a \cdot b \cdot (\sup \cdot (\mathsf{w} \cdot a \cdot b) \cdot c \cdot d) \in \mathcal{T}$ and $b \cdot c \cdot e \in \mathcal{T}$ then $\mathsf{d} \cdot (\sup \cdot (\mathsf{w} \cdot a \cdot b) \cdot c \cdot d) \cdot e = d \cdot e$,
18. if $\mathsf{x} \cdot a \cdot b \in \mathcal{T}$ and $\mathsf{f} \cdot a \cdot \mathsf{h} \cdot b \in \mathcal{T}$ then $b \cdot (\epsilon \cdot a \cdot b) \in \mathcal{T}$,
19. if $\mathsf{x} \cdot a \cdot a \in \mathcal{T}$ and $\mathsf{f} \cdot a \cdot \mathsf{h} \cdot b \in \mathcal{T}$ then $a \cdot (\epsilon \cdot a \cdot b) \in \mathcal{T}$,
20. if $a \cdot c \in \mathcal{T}$, $\mathsf{f} \cdot (b \cdot c) \cdot (\mathsf{w} \cdot a \cdot b) \cdot d \in \mathcal{T}$ and $\mathsf{L} \cdot (\mathsf{w} \cdot a \cdot b) \in \mathcal{T}$ then $\mathsf{w} \cdot a \cdot b \cdot (\sup \cdot c \cdot d) \in \mathcal{T}$,
21. if $\mathsf{L} \cdot a \in \mathcal{T}$ and for every $c \in \mathcal{T}(a)$ we have $\mathsf{L} \cdot (b \cdot c) \in \mathcal{T}$, then $\mathsf{L} \cdot (\mathsf{g} \cdot a \cdot b) \in \mathcal{T}$,
22. if $\mathsf{L} \cdot a \in \mathcal{T}$ and $\mathsf{f} \cdot a \cdot \mathsf{L} \cdot b \in \mathcal{T}$ then $\mathsf{L} \cdot (\mathsf{\Sigma} \cdot a \cdot b) \in \mathcal{T}$,
23. if $\mathsf{L} \cdot a \in \mathcal{T}$ and $\mathsf{f} \cdot a \cdot \mathsf{L} \cdot b \in \mathcal{T}$ then $\mathsf{L} \cdot (\mathsf{w} \cdot a \cdot b) \in \mathcal{T}$,
24. if $\mathsf{L} \cdot a \in \mathcal{T}$ and $\mathsf{f} \cdot a \cdot \mathsf{h} \cdot b \in \mathcal{T}$ then $\mathsf{L} \cdot (\upsilon \cdot a \cdot b) \in \mathcal{T}$,
25. if $\mathsf{L} \cdot (\mathsf{w} \cdot a \cdot b) \in \mathcal{T}$ and for every $c \in \mathcal{C}$ such that $a \cdot c \in \mathcal{T}$, $\mathsf{f} \cdot (b \cdot c) \cdot (\mathsf{w} \cdot a \cdot b) \cdot d \in \mathcal{T}$ and $\mathsf{\Xi} \cdot (b \cdot c) \cdot (\mathsf{s} \cdot (\mathsf{k} \cdot e) \cdot d) \in \mathcal{T}$ we have $e \cdot (\sup \cdot (\mathsf{w} \cdot a \cdot b) \cdot c \cdot d) \in \mathcal{T}$, then $\mathsf{\Xi} \cdot (\mathsf{w} \cdot a \cdot b) \cdot e \in \mathcal{T}$,
26. if $\mathsf{L} \cdot a \in \mathcal{T}$ and for every $e \in \mathcal{T}(a)$ we have $\mathsf{q} \cdot (b \cdot e) \cdot (c \cdot e) \cdot (d \cdot e) \in \mathcal{T}$, then $\mathsf{q} \cdot (\mathsf{g} \cdot a \cdot b) \cdot c \cdot d \in \mathcal{T}$,
27. if $a, b \in \mathcal{T}$ or $a, b \in \mathcal{F}$ then $\mathsf{q} \cdot \mathsf{h} \cdot a \cdot b \in \mathcal{T}$,
28. if $\mathsf{q} \cdot a \cdot (\boldsymbol{\pi}_1 \cdot c) \cdot (\boldsymbol{\pi}_1 \cdot d) \in \mathcal{T}$, $\mathsf{q} \cdot (b \cdot (\boldsymbol{\pi}_1 \cdot c)) \cdot (\boldsymbol{\pi}_2 \cdot c) \cdot (\boldsymbol{\pi}_2 \cdot d) \in \mathcal{T}$ and $\mathsf{L} \cdot (\mathsf{\Sigma} \cdot a \cdot b) \in \mathcal{T}$ then $\mathsf{q} \cdot (\mathsf{\Sigma} \cdot a \cdot b) \cdot c \cdot d \in \mathcal{T}$,
29. if $\mathsf{q} \cdot a \cdot c \cdot c' \in \mathcal{T}$, $\mathsf{q} \cdot (\mathsf{f} \cdot (b \cdot c) \cdot (\mathsf{w} \cdot a \cdot b)) \cdot d \cdot d' \in \mathcal{T}$ and $\mathsf{L} \cdot (\mathsf{w} \cdot a \cdot b) \in \mathcal{T}$ then $\sup \cdot (\mathsf{w} \cdot a \cdot b) \cdot c \cdot d = \sup \cdot (\mathsf{w} \cdot a \cdot b) \cdot c' \cdot d'$.

Let $\mathcal{M}$ be an $\mathcal{I}^+$-model. An $\mathcal{M}$-*valuation* is a function from the set of variables V to $\mathcal{C}$ (cf. Definition 2.3.17). Given an $\mathcal{M}$-valuation $\rho : V \to \mathcal{C}$ we define the *value* of $M \in \mathbb{T}_{\mathrm{CL}}$, denoted $[\![M]\!]^{\mathcal{M}}_\rho$ or just $[\![M]\!]_\rho$, by induction on the structure of M:

- $[\![x]\!]_\rho = \rho(x)$ if $x \in V$,
- $[\![\mathsf{K}]\!]_\rho = \mathsf{k}$, $[\![\mathsf{S}]\!]_\rho = \mathsf{s}$,
- $[\![c]\!]_\rho = I(c)$ if $c \in \Sigma$,
- $[\![M_1 M_2]\!]_\rho = [\![M_1]\!]_\rho \cdot [\![M_1]\!]_\rho$.

If $[\![M]\!]^{\mathcal{M}}_\rho \in \mathcal{T}$, we write $\mathcal{M}, \rho \models M$. If M is closed then we write $\mathcal{M} \models M$. We write $\mathcal{M}, \rho \models \Gamma$ if $\mathcal{M}, \rho \models M$ for all $M \in \Gamma$. We write $\Gamma \models_{\mathcal{I}^+} M$ if for every $\mathcal{I}^+$-model $\mathcal{M}$ and every $\mathcal{M}$-valuation ρ, the condition $\mathcal{M}, \rho \models \Gamma$ implies $\mathcal{M}, \rho \models M$. We use the notation $\Gamma \models_{\mathcal{I}^+} M_1 = M_2$ if for every $\mathcal{I}^+$-model $\mathcal{M}$ and every $\mathcal{M}$-valuation ρ, the condition $\mathcal{M}, \rho \models \Gamma$ implies $[\![M_1]\!]^{\mathcal{M}}_\rho = [\![M_2]\!]^{\mathcal{M}}_\rho$.

Theorem 7.1.9. *If $\Gamma \vdash_{\mathcal{I}^+} X$ then $\Gamma \models_{\mathcal{I}^+} X$. Also, if $\Gamma \vdash_{\mathcal{I}^+} X = Y$ then $\Gamma \models_{\mathcal{I}^+} X = Y$.*

Proof. Induction on the length of derivation. □

7.2 Model construction

In this section we construct a model for $\mathcal{I}^+$. This implies the consistency of $\mathcal{I}^+$. We assume the existence of a strongly inaccessible cardinal, i.e., in this section we work in ZFC+SI (see Section 2.2). The existence of a strongly inaccessible cardinal is necessary to handle dependent function types, dependent sums and W-types. Without a strongly inaccessible cardinal we just would not be able to define the set of types $\mathscr{T}$.

Definition 7.2.1. The set of types $\mathscr{T}$ is defined by a fixpoint construction. We define $\mathscr{T}_\alpha$ by induction on an ordinal α, together with the domains $\mathcal{D}_\tau$. As usual, we set $\mathscr{T}_{<\alpha} = \bigcup_{\beta<\alpha} \mathscr{T}_\beta$.

- $\varepsilon, o \in \mathscr{T}_\alpha$, and $\mathcal{D}_\varepsilon = \emptyset$, $\mathcal{D}_o = \{\top, \bot\}$,
- if $\tau \in \mathscr{T}_{<\alpha}$ and $S \subseteq \mathcal{D}_\tau$ then[1] $\Upsilon(\tau, S) \in \mathscr{T}_\alpha$ and $\mathcal{D}_{\Upsilon(\tau,S)} = S$,
- if $\tau \in \mathscr{T}_{<\alpha}$ and F is a function from $\mathcal{D}_\tau$ to $\mathscr{T}_{<\alpha}$ then $\mathsf{G}(\tau, F) \in \mathscr{T}_\alpha$ and
$$\mathcal{D}_{\mathsf{G}(\tau,F)} = \prod_{d\in\mathcal{D}_\tau} \mathcal{D}_{F(d)},$$
- if $\tau \in \mathscr{T}_{<\alpha}$ and F is a function from $\mathcal{D}_\tau$ to $\mathscr{T}_{<\alpha}$ then $\Sigma(\tau, F) \in \mathscr{T}_\alpha$ and
$$\mathcal{D}_{\Sigma(\tau,F)} = \{\langle d_1, d_2\rangle \mid d_1 \in \mathcal{D}_\tau, d_2 \in \mathcal{D}_{F(d_1)}\},$$
- if $\tau \in \mathscr{T}_{<\alpha}$ and F is a function from $\mathcal{D}_\tau$ to $\mathscr{T}_{<\alpha}$ such that there is $d \in \mathcal{D}_\tau$ with $\mathcal{D}_{F(d)} = \emptyset$, then $\mathsf{W}(\tau, F) \in \mathscr{T}_\alpha$ and $\mathcal{D}_{\mathsf{W}(\tau,F)}$ is defined as follows. Let γ be the maximum of ω and the least cardinal greater than the cardinality of $\bigcup_{d\in\mathcal{D}_\tau} \mathcal{D}_{F(d)}$. We define $\mathcal{D}^\beta_{\mathsf{W}(\tau,F)}$ inductively by:
  - if $d \in \mathcal{D}_\tau$ and f is a function from $\mathcal{D}_{F(d)}$ to $\mathcal{D}^{<\beta}_{\mathsf{W}(\tau,F)} = \bigcup_{\delta<\beta} \mathcal{D}^\delta_{\mathsf{W}(\tau,F)}$ then
  $$\langle d, f\rangle \in \mathcal{D}^\beta_{\mathsf{W}(\tau,F)}.$$

  We set $\mathcal{D}_{\mathsf{W}(\tau,F)} = \bigcup_{\beta<\gamma} \mathcal{D}^\beta_{\mathsf{W}(\tau,F)}$.

Because there exists a strongly inaccessible cardinal, by Lemma 2.2.6 there is a Grothendieck universe U. Using Lemma 2.2.5 one shows by induction on α that $\mathscr{T}_\alpha \subseteq U$ and $\mathcal{D}_\tau \in U$ for $\tau \in \mathscr{T}_\alpha$. It is also easy to see that $\mathscr{T}_\alpha \subseteq \mathscr{T}_\beta$ for $\alpha \le \beta$. Hence we can apply Theorem 2.1.3 to obtain an ordinal ζ with $\mathscr{T}_\zeta = \mathscr{T}_{<\zeta}$. We take $\mathscr{T} = \mathscr{T}_\zeta$.

If F is a function from $\mathcal{D}_{\tau_1}$ to $\mathscr{T}$ such that $F(d) = \tau_2$ for every $d \in \mathcal{D}_{\tau_1}$, for some fixed $\tau_2 \in \mathscr{T}$, then instead of $\mathsf{G}(\tau_1, F)$ we also write $\tau_1 \to \tau_2$. If $F(d) = \tau_2$ for all $d \in \mathcal{D}_{\tau_1}$ then we use the abbreviation $\tau_1 \times \tau_2 = \Sigma(\tau_1, F)$.

The set of terms $\mathbb{T}$ is defined to be the set of all combinatory terms over the signature containing all constants of $\mathcal{I}^+$ plus a distinct constant d^τ for each $d \in \mathcal{D}_\tau$ with $\tau = o$ or

[1] Formally, $\Upsilon(\tau, S)$ should be understood as a triple $\langle \Upsilon, \tau, S\rangle$ where Υ is some "tag" (some appropriately constructed constant set) uniquely identifying the kind of this type (i.e. the tag signifies that this is a subtype).

$\tau = \mathsf{G}(\tau', F) \in \mathscr{T}$ or $\tau = \Sigma(\tau', F) \in \mathscr{T}$ or $\tau = \mathsf{W}(\tau', F) \in \mathscr{T}$. To save on notation, we usually drop the superscript τ, i.e., we confuse elements of $\mathcal{D}_\tau$ with the corresponding constants in $\mathbb{T}$. The point of superscripting elements with their types is to ensure that each constant f has a uniquely determined type which is not a subtype. Note that, e.g., the constants corresponding to the elements of $\mathcal{D}_{\Upsilon(o,S)}$ are superscripted with the type o, not with $\Upsilon(o, S)$.

If $f \in \mathcal{D}_{\mathsf{G}(\tau,F)}$ then we use the notation $f^{\mathcal{F}}(d)$ for the value of f at $d \in \mathcal{D}_\tau$, to avoid confusion with the term fd (i.e. with $f^{\mathsf{G}(\tau,F)} d^\tau$).

Lemma 7.2.2. *$\langle d, f\rangle \in \mathcal{D}_{\mathsf{W}(\tau,F)}$ iff $d \in \mathcal{D}_\tau$ and f is a function from $\mathcal{D}_{F(d)}$ to $\mathcal{D}_{\mathsf{W}(\tau,F)}$.*

Proof. The implication from left to right follows directly from definitions. For the other direction, assume that $d \in \mathcal{D}_\tau$ and f is a function from $\mathcal{D}_{F(d)}$ to $\mathcal{D}_{\mathsf{W}(\tau,F)} = \bigcup_{\alpha<\gamma} \mathcal{D}^\alpha_{\mathsf{W}(\tau,F)}$. Then for every $e \in \mathcal{D}_{F(d)}$ there is $\alpha(e) < \gamma$ with $f(e) \in \mathcal{D}^{\alpha(e)}_{\mathsf{W}(\tau,F)}$. It suffices to show that $\sup_{e\in\mathcal{D}_{F(d)}} \alpha(e) < \gamma$. First assume $\gamma > \omega$. Note that each $\alpha(e)$ has cardinality at most $\mu = |\bigcup_{d\in\mathcal{D}_\tau} \mathcal{D}_{F(d)}|$, because $|\alpha(e)| \le \alpha(e) < \gamma$, and γ is the least cardinal greater than μ. Hence we have

$$\begin{array}{rcl} |\sup_{e\in\mathcal{D}_{F(d)}} \alpha(e)| & = & |\bigcup_{e\in\mathcal{D}_{F(d)}} \alpha(e)| \\ & \le & |\mathcal{D}_{F(d)}|\mu \\ & \le & \mu^2. \end{array}$$

Because $\gamma > \omega$ and γ is the least cardinal greater than μ, the cardinal μ is infinite and we have $\mu = \mu^2$. Therefore $|\sup_{e\in\mathcal{D}_{F(d)}} \alpha(e)| \le \mu^2 = \mu < \gamma$. Since γ is a cardinal, this implies $\sup_{e\in\mathcal{D}_{F(d)}} \alpha(e) < \gamma$. If $\gamma = \omega$ then $\bigcup_{d\in\mathcal{D}_\tau} \mathcal{D}_{F(d)}$ is finite. Hence so is $\mathcal{D}_{F(d)}$ and thus $\sup_{e\in\mathcal{D}_{F(d)}} \alpha(e)$ is also finite, i.e., $\sup_{e\in\mathcal{D}_{F(d)}} \alpha(e) < \omega = \gamma$. □

In this section we adopt the following conventions:

- $\mathsf{L}X \equiv \Xi XX$,
- $\mathsf{G}XY \equiv \lambda f.\Xi X(\lambda x.Yx(fx))$ where $x \notin \mathrm{FV}(Y)$ and $f \notin \mathrm{FV}(X, Y)$, i.e.,

$$\mathsf{G}XY \equiv \mathsf{S}(\mathsf{K}(\Xi X))(\mathsf{S}(\mathsf{S}(\mathsf{S}(\mathsf{K}Y)\mathsf{I}))(\mathsf{S}(\mathsf{S}(\mathsf{KS})(\mathsf{S}(\mathsf{KK})\mathsf{I}))(\mathsf{KI}))),$$

- $\pi XY \equiv \lambda x.XY$,
- $\pi_1 X \equiv X\mathsf{K}$,
- $\pi_2 X \equiv X(\mathsf{KI})$,
- $\Upsilon AX \equiv \lambda x.Ax \wedge Xx$,
- $\Sigma AB \equiv \lambda x.A(\pi_1 x) \wedge B(\pi_1 x)(\pi_2 x)$.

In other words, whenever we write, e.g., $\pi_1 X$, this denotes the term $X\mathsf{K}$, not the term $(\lambda x.x\mathsf{K})X$. This convention allows us to shorten some notations. Its significance is purely technical. Without it we would simply have to replace, e.g., $\pi_1 X$ with $X\mathsf{K}$ in some places below. The important thing is that $\mathsf{L}X$, $\mathsf{G}XY$, πXY, ΥAX, ΣAB are never a w-redex, and $\pi_1 c$, $\pi_2 c$ are not a w-redex when c is a constant.

Definition 7.2.3. For $\tau \in \mathscr{T}$ and an ordinal α we define the representation relations $\succ_\tau^\alpha \in \mathbb{T} \times \mathbb{T}$, the contraction relation $\to^\alpha \in \mathbb{T} \times \mathbb{T}$, and the relation $\succ_{\mathscr{T}}^\alpha \in \mathbb{T} \times \mathscr{T}$ inductively. The notation $X \rightsquigarrow_\tau^\alpha Y$ stands for $X \overset{*}{\to}^\alpha \cdot \succ_\tau^\alpha Y$, and the notations $\succ_\tau^{<\alpha}$, $\rightsquigarrow_\tau^{<\alpha}$ are defined as usual. Let η_τ be a choice function for $\mathbb{P}(\mathcal{D}_\tau) \setminus \{\emptyset\}$ and let $D_{\tau,p} = \{d \in \mathcal{D}_\tau \mid p^{\mathcal{F}}(d) = \top\}$ for $p \in \mathcal{D}_{\tau \to o}$.

(w_1) $\mathsf{K}XY \to^\alpha X$,

(w_2) $\mathsf{S}XYZ \to^\alpha XZ(YZ)$,

(γ) $fX \to^\alpha b$ if $f \in \mathcal{D}_{\mathsf{G}(\tau,F)}$, $f^{\mathcal{F}}(a) = b$ and $X \succ_\tau^{<\alpha} a$, for some $a \in \mathcal{D}_\tau$,

(ϵ_1) $\epsilon AX \to^\alpha \eta_\tau(D_{\tau,p})$ if $A \succ_{\mathscr{T}}^{<\alpha} \tau$, $X \succ_{\tau\to o}^{<\alpha} p$ and $D_{\tau,p} \neq \emptyset$,

(ϵ_2) $\epsilon AX \to^\alpha \eta_\tau(\mathcal{D}_\tau)$ if $A \succ_{\mathscr{T}}^{<\alpha} \tau$, $\mathcal{D}_\tau \neq \emptyset$, $X \succ_{\tau\to o}^{<\alpha} p$ and $D_{\tau,p} = \emptyset$,

(μ_1) $\mathsf{M}X \to^\alpha \mathsf{K}$ if $X \succ_o^{<\alpha} \top$,

(μ_2) $\mathsf{M}X \to^\alpha \mathsf{KI}$ if $X \succ_o^{<\alpha} \bot$,

(π_1) $\pi_1\langle a,b\rangle^\tau \to^\alpha a$ if $\tau = \Sigma(\tau', F)$,

(π_2) $\pi_2\langle a,b\rangle^\tau \to^\alpha b$ if $\tau = \Sigma(\tau', F)$,

(sup) $\sup AXY \to^\alpha \langle d,f\rangle^\tau$ if $A \succ_{\mathscr{T}}^{<\alpha} \tau = \mathsf{W}(\tau', F)$, $X \succ_{\tau'}^{<\alpha} d$ and $Y \succ_{F(d)\to\tau}^{<\alpha} f$,

(D) $\mathsf{D}\langle d,f\rangle^\tau X \to^\alpha fX$ if $\tau = \mathsf{W}(\tau', F)$,

($\mathcal{D}_\tau$) $d \succ_\tau^\alpha d$ if $d \in \mathcal{D}_\tau$ and $\tau = o$ or $\tau = \mathsf{W}(\tau', F)$,

($\mathcal{F}_\tau$) $X \succ_\tau^\alpha d$ if $\tau = \mathsf{G}(\tau', F)$, $d \in \mathcal{D}_\tau$ and for every $a \in \mathcal{D}_{\tau'}$ we have $Xa \rightsquigarrow_{F(a)}^{<\alpha} d^{\mathcal{F}}(a)$,

($\mathcal{S}_\tau$) $X \succ_\tau^\alpha d$ if $\tau = \Upsilon(\tau', S)$, $X \succ_{\tau'}^{<\alpha} d$ and $d \in S$,

(π_τ^Σ) $X \succ_\tau^\alpha \langle a,b\rangle^\tau$ if $\tau = \Sigma(\tau', F)$, $a \in \mathcal{D}_{\tau'}$, $b \in \mathcal{D}_{F(a)}$, and $\pi_1 X \rightsquigarrow_{\tau'}^{<\alpha} a$ and $\pi_2 X \rightsquigarrow_{F(a)}^{<\alpha} b$,

($\neg_\top$) $\neg X \succ_o^\alpha \top$ if $X \succ_o^{<\alpha} \bot$,

($\neg_\bot$) $\neg X \succ_o^\alpha \bot$ if $X \succ_o^{<\alpha} \top$,

($\mathsf{V}_\top$) $X \vee Y \succ_o^\alpha \top$ if $X \succ_o^{<\alpha} \top$ or $Y \succ_o^{<\alpha} \top$,

($\mathsf{V}_\bot$) $X \vee Y \succ_o^\alpha \bot$ if $X \succ_o^{<\alpha} \bot$ and $Y \succ_o^{<\alpha} \bot$,

($\mathsf{\Lambda}_\top$) $X \wedge Y \succ_o^\alpha \top$ if $X \succ_o^{<\alpha} \top$ and $Y \succ_o^{<\alpha} \top$,

($\mathsf{\Lambda}_\bot$) $X \wedge Y \succ_o^\alpha \bot$ if $X \succ_o^{<\alpha} \bot$ or $Y \succ_o^{<\alpha} \bot$,

($\Xi_\top$) $\Xi XY \succ_o^\alpha \top$ if $X \succ_{\mathscr{T}}^{<\alpha} \tau$ and for every $d \in \mathcal{D}_\tau$ we have $Yd \rightsquigarrow_o^{<\alpha} \top$,

($\Xi_\bot$) $\Xi XY \succ_o^\alpha \bot$ if $X \succ_{\mathscr{T}}^{<\alpha} \tau$ and there exists $d \in \mathcal{D}_\tau$ with $Yd \rightsquigarrow_o^{<\alpha} \bot$,

($\mathsf{W}_\top$) $\mathsf{W}AB\langle d,f\rangle^\tau \succ_o^\alpha \top$ if $\mathsf{W}AB \succ_{\mathscr{T}}^{<\alpha} \tau$,

($\mathsf{T}_\top$) $\mathsf{T}\langle d,f\rangle^\tau X \succ_o^\alpha \top$ if $\tau = \mathsf{W}(\tau', F)$ and $X \succ_{\tau'}^{<\alpha} d$

($\mathsf{T}_\bot$) $\mathsf{T}\langle d,f\rangle^\tau X \succ_o^\alpha \bot$ if $\tau = \mathsf{W}(\tau', F)$, $X \succ_{\tau'}^{<\alpha} d'$ and $d' \not\equiv d$,

($\mathsf{L}_\top$) $\mathsf{L}X \succ_o^\alpha \top$ if $X \succ_{\mathscr{T}}^{<\alpha} \tau$ for some $\tau \in \mathscr{T}$,

($\mathsf{H}_{\mathscr{T}}$) $\mathsf{H} \succ_{\mathscr{T}}^\alpha o$,

($\mathsf{O}_{\mathscr{T}}$) $\mathsf{O} \succ^{\alpha}_{\mathscr{T}} \varepsilon$,

($\mathsf{G}_{\mathscr{T}}$) $\mathsf{G}AB \succ^{\alpha}_{\mathscr{T}} \mathsf{G}(\tau, F)$ if $A \succ^{<\alpha}_{\mathscr{T}} \tau$ and for every $d \in \mathcal{D}_\tau$ we have $Bd \rightsquigarrow^{<\alpha}_{\mathscr{T}} F(d)$,

($\Sigma_{\mathscr{T}}$) $\Sigma AB \succ^{\alpha}_{\mathscr{T}} \Sigma(\tau, F)$ if $A \succ^{<\alpha}_{\mathscr{T}} \tau$ and for every $d \in \mathcal{D}_\tau$ we have $Bd \rightsquigarrow^{<\alpha}_{\mathscr{T}} F(d)$,

($\mathsf{W}_{\mathscr{T}}$) $\mathsf{W}AB \succ^{\alpha}_{\mathscr{T}} \mathsf{W}(\tau, F)$ if $A \succ^{<\alpha}_{\mathscr{T}} \tau$ and for every $d \in \mathcal{D}_\tau$ we have $Bd \rightsquigarrow^{<\alpha}_{\mathscr{T}} F(d)$,

($\Upsilon_{\mathscr{T}}$) $\Upsilon AX \succ^{\alpha}_{\mathscr{T}} \Upsilon(\tau, S_p)$ if $A \succ^{<\alpha}_{\mathscr{T}} \tau$, $X \succ^{<\alpha}_{\tau \to o} p$ and

$$S_p = \{d \in \mathcal{D}_\tau \mid p^{\mathcal{F}}(d) \equiv \top\}.$$

It is to be understood that the relation $\to^{\alpha}$ is the compatible closure of the rules (w_1), (w_2), (γ), (ϵ_1), (ϵ_2), (μ_1), (μ_2), (π_1), (π_2), (sup) and (D), while the relations $\succ^{\alpha}_{\tau}$ for $\tau \in \mathscr{T}$ and $\succ^{\alpha}_{\mathscr{T}}$ are defined directly by the corresponding rules, i.e., without taking compatible closure – these are not contraction relations.

It is easy to see that for $\alpha \le \kappa$ we have $\to^{\alpha} \subseteq \to^{\kappa}$, $\succ^{\alpha}_{\tau} \subseteq \succ^{\kappa}_{\tau}$ for $\tau \in \mathscr{T}$, and $\succ^{\alpha}_{\mathscr{T}} \subseteq \succ^{\kappa}_{\mathscr{T}}$. Hence by Theorem 2.1.3 there is the closure ordinal ζ with $\to^{\zeta} = \to^{<\zeta}$, $\succ^{\zeta}_{\tau} = \succ^{<\zeta}_{\tau}$ for $\tau \in \mathscr{T}$, and $\succ^{\zeta}_{\mathscr{T}} = \succ^{<\zeta}_{\mathscr{T}}$. We use the notations $\to$, $\succ_\tau$ ($\tau \in \mathscr{T}$), $\succ_{\mathscr{T}}$ for $\to^{\zeta}$, $\succ^{\zeta}_{\tau}$ ($\tau \in \mathscr{T}$), $\succ^{\zeta}_{\mathscr{T}}$, respectively.

By $\to_\gamma$ we denote the γ-*contraction* relation determined by the rule (γ), by $\to_\epsilon$ the ϵ-*contraction* relation determined by the rules (ϵ_1) and (ϵ_2), and so on. We also use the notations $\to^{\alpha}_{\gamma}$, $\to^{\alpha}_{\epsilon}$, etc., accordingly.

We define the reduction system R by $R = \langle \to, \{\succ_\tau\}_{\tau \in \mathscr{T}} \cup \{\succ_{\mathscr{T}}\}\rangle$. The reduction system R^α is defined by $R^\alpha = \langle \to^\alpha, \{\succ^{\alpha}_{\tau}\}_{\tau \in \mathscr{T}} \cup \{\succ^{\alpha}_{\mathscr{T}}\}\rangle$.

The following lemma will often be used implicitly. Note that this lemma would be false in the lambda-calculus with β-reduction.

Lemma 7.2.4. *If* $\mathsf{G}XY \stackrel{*}{\to} Z$ *then* $Z \equiv \mathsf{G}X'Y'$ *with* $X \stackrel{*}{\to} X'$ *and* $Y \stackrel{*}{\to} Y'$. *An analogous result holds when* $\Sigma XY \stackrel{*}{\to} Z$ *or* $\Upsilon XY \stackrel{*}{\to} Z$ *or* $\mathsf{W}XY \stackrel{*}{\to} Z$. *Here* $\stackrel{*}{\to}$ *may be any of* $\stackrel{*}{\to}_w$, $\stackrel{*}{\to}^{\alpha}$, $\stackrel{*}{\to}_\gamma$, *etc.*

The general strategy of the correctness proof for the model construction is the same as in Section 6.2, we just need to consider the additional cases for the new illative primitives. We show that the reduction system R is coherent and invariant, and then we use these properties to show a sequence of lemmas corresponding to the conditions in the definition of an $\mathcal{I}^+$-model (see Definition 7.1.8). Because $\mathcal{I}^+$ is based on combinatory logic with weak equality, no reduction "below lambdas" is possible, so in contrast to Section 6.2 it is not necessary to prove that R is closed under substitution. The fact that $\mathcal{I}^+$ is based on combinatory logic with weak equality also allows us to avoid many purely technical problems which would otherwise appear.

Lemma 7.2.5. *For all ordinals* α, κ *the reduction systems* R^α *and* R^κ *are mutually coherent. In particular, the reduction system* R *is coherent.*

Proof. Like in Lemma 6.2.6, we proceed by induction on pairs of ordinals $\langle \alpha, \kappa \rangle$ ordered componentwise. We need to show the conditions:

(a) $\to^\alpha$ and $\to^\kappa$ commute,

(b) $\to^\kappa$ preserves $\succ_i^\alpha$,

(c) $\to^\alpha$ preserves $\succ_i^\kappa$,

(d) if $X \succ_i^\alpha d_1$ and $X \succ_i^\kappa d_2$ then $d_1 = d_2$,

where $i \in \mathscr{T}$ or $i = \mathscr{T}$.

So assume $(a)-(d)$ hold for all pairs of ordinals $\langle \alpha', \kappa' \rangle$ with $\alpha' < \alpha$ and $\kappa' \le \kappa$, or $\alpha' \le \alpha$ and $\kappa' < \kappa$. We show that $(a) - (d)$ also hold for $\langle \alpha, \kappa \rangle$.

As in Lemma 6.2.6 one may show the following two conditions $(\star)$ and $(\star\star)$, by identical proofs.

$(\star)$ If $X \rightsquigarrow_i^{<\alpha} d$ and $X \to^\kappa Y$ then $Y \rightsquigarrow_i^{<\alpha} d$, where $i \in \mathscr{T}$ or $i = \mathscr{T}$. The same holds with α and κ exchanged.

$(\star\star)$ If $X \rightsquigarrow_i^{<\alpha} d_1$ and $X \rightsquigarrow_i^\kappa d_2$ then $d_1 = d_2$. The same holds with α and κ exchanged.

Now we prove $(a) - (d)$.

(a) Define $\to_s^\nu = \to_\gamma^\nu \cup \to_\epsilon^\nu \cup \to_\mu^\nu \cup \to_\pi \cup \to_{\mathrm{sup}}^\nu \cup \to_{\mathsf{D}}^\nu$ for an ordinal ν. We show that the following pairs of relations commute: $\to_w$ and $\to_s^\alpha$, $\to_w$ and $\to_s^\kappa$, $\to_s^\alpha$ and $\to_s^\kappa$. Since $\to_w$ is confluent, $\to^\alpha = \to_w \cup \to_s^\alpha$ and $\to^\kappa = \to_w \cup \to_s^\kappa$, it then follows from the general Hindley-Rosen Lemma 2.3.3 that $\to^\alpha$ and $\to^\kappa$ commute.

First we show that $\to_s^\alpha$ and $\to_s^\kappa$ commute. Assume $X \to_s^\alpha X_1$ and $X \to_s^\kappa X_2$. We show that there is X' with $X_1 \overset{\equiv}{\to}{}_s^\kappa X' \overset{\equiv}{\leftarrow}{}_s^\alpha X_2$. Without loss of generality assume that the contraction $X \to_s^\alpha X_1$ occurs at the root. We consider possible rules by which the contraction $X \to_s^\alpha X_1$ may occur.

(γ) We have $X \equiv fY$, $f \in \mathcal{D}_{\mathsf{G}(\tau,F)}$, $X_1 \equiv f^{\mathcal{F}}(d_1)$ and $Y \succ_\tau^{<\alpha} d_1$. If the contraction $X \equiv fY \to_s^\kappa X_2$ also occurs at the root then it is a γ-contraction and we have $X_2 \equiv f^{\mathcal{F}}(d_2)$ and $Y \succ_\tau^{<\kappa} d_2$. By part (d) of the IH we obtain $d_1 = d_2$, so we may take $X' \equiv X_1 \equiv X_2$. Otherwise, $X_2 \equiv fY'$ with $Y \to_s^\kappa Y'$. Since $Y \succ_\tau^{<\alpha} d_1$, by part (b) of the IH we have $Y' \succ_\tau^{<\alpha} d_1$. Thus still $X_2 \equiv fY' \to_\gamma^\alpha d_1 \equiv X_1$, so we may take $X' \equiv X_1$.

(ϵ_1) We have $X \equiv \epsilon AY$, $A \succ_{\mathscr{T}}^{<\alpha} \tau$, $Y \succ_{\tau\to o}^{<\alpha} p$, and $X_1 \equiv d$ for appropriate $d \in \mathcal{D}_\tau$. If the contraction $\epsilon AY \to_\epsilon^\kappa X_2$ also occurs at the root, then $A \succ_{\mathscr{T}}^{<\alpha} \tau'$, $Y \succ_{\tau'\to o}^{<\alpha} p'$ and $X_2 \equiv d'$ for appropriate $d' \in \mathcal{D}_{\tau'}$. By part (d) of the IH we conclude that $\tau' = \tau$ and $p' \equiv p$, which implies $d' \equiv d$. Otherwise, if the contraction $X \to_s^\kappa X_2$ does not occur at the root, then $X_2 \equiv \epsilon A'Y'$ with $A \overset{\equiv}{\to}{}_s^\kappa A'$ and $Y \overset{\equiv}{\to}{}_s^\beta X'$. Using part (b) of the IH one checks that still $X_2 \equiv \epsilon A'Y' \to_\epsilon^\alpha d$.

(ϵ_2) The argument is analogous to the case for (ϵ_1).

(μ_1) We have $X \equiv \mathsf{M}Y$, $X_1 \equiv \mathsf{K}$ and $Y \succ_o^{<\alpha} \top$. If the contraction $X \to_s^\kappa X_2$ occurs at the root, then it is a μ-contraction, and either $X_2 \equiv \mathsf{K}$, in which case we may take $X' \equiv \mathsf{K}$, or $X_2 \equiv \mathsf{KI}$. But if $X_2 \equiv \mathsf{KI}$ then $Y \succ_o^{<\alpha} \bot$ which contradicts part (d) of the IH. If the contraction $X \to_s^\kappa X_2$ does not occur at the root, then

$X_2 \equiv \mathsf{M}Y'$ with $Y \to_s^\kappa Y'$. Then $Y' \succ_o^{<\alpha} \top$ by part (b) of the IH. Thus still $X_2 \equiv \mathsf{M}Y' \to_\mu^\alpha \mathsf{K} \equiv X_1$, so we may take $X' \equiv X_1$.

(μ_2) Analogous to the case for (μ_1).

(π_1) We have $X \equiv \pi_1\langle a,b\rangle$ and $X_1 \equiv a$. By inspecting definitions one sees that $X \to_s^\kappa X_2$ is only possible when $X_2 \equiv a$.

(π_2) Analogous to the case for (π_1).

(sup) We have $X \equiv \sup AZ_1Z_2$, $X_1 \equiv \langle d,f\rangle^\tau$, $A \succ_{\mathscr{T}}^{<\alpha} \tau = \mathsf{W}(\tau',F)$, $Z_1 \succ_{\tau'}^{<\alpha} d$ and $Z_2 \succ_{F(d)\to\tau}^{<\alpha} f$. If the contraction $X \to_s^\kappa X_2$ occurs at the root, then it is a sup-contraction, and $X_2 \equiv \langle d_0,f_0\rangle^{\tau_0}$ $A \succ_{\mathscr{T}}^{<\kappa} \tau_0 = \mathsf{W}(\tau_0',F_0)$, $Z_1 \succ_{\tau_0'}^{<\kappa} d_0$ and $Z_2 \succ_{F_0(d_0)\to\tau_0}^{<\kappa} f_0$. Using part (d) of the IH we then conclude that $\tau_0 = \tau$, $d_0 = d$ and $f_0 = f$, so $X_2 \equiv X_1$. If the contraction $X \to_s^\kappa X_2$ does not occur at the root, then $X_2 \equiv \sup A'Z_1'Z_2'$ with $A \overset{\equiv}{\to}{}_s^\kappa A'$, $Z_1 \overset{\equiv}{\to}{}_s^\kappa Z_1'$, $Z_2 \overset{\equiv}{\to}{}_s^\kappa Z_2'$. Using part (b) of the IH we conclude that $X_2 \to_{\mathrm{sup}}^\alpha X_1$.

(D) We have $X \equiv \mathsf{D}\langle d,f\rangle Y$ and $X_1 \equiv fY$. If the contraction $X \to_s^\kappa X_2$ occurs at the root, then $X_2 \equiv X_1$. Otherwise, $X_2 \equiv \mathsf{D}\langle d,f\rangle Y'$ where $Y \to_s^\kappa Y'$. Then $X_2 \to_s^\alpha fY'$ and $X_1 \to_s^\kappa fY'$.

Now we show that $\to_w$ and $\to_s^\alpha$ commute. The proof for $\to_w$ and $\to_s^\kappa$ is completely analogous. We show that if $X \to_s^\alpha X_1$ and $X \to_w X_2$ then there is X' such that $X_1 \to_w X'$ and $X_2 \overset{*}{\to}{}_s^\alpha X'$. Then it will follow from Lemma 2.3.4 that $\to_w$ and $\to_s^\alpha$ commute. So assume $X \to_s^\alpha X_1$ and $X \to_w X_2$. If the contraction $X \to_s^\alpha X_1$ is at the root, then the proof is analogous to an appropriate case considered above. For instance, if the contraction $X \to_s^\alpha X_1$ is a γ-contraction, then $X \equiv fY$ for some $f \in \mathcal{D}_{\mathsf{G}(\tau,F)}$, $Y \succ_\tau^{<\alpha} d$ and $X_1 \equiv f^{\mathcal{F}}(d)$. Hence $X_2 \equiv fY'$ with $Y \to_w Y'$. By part (b) of the IH we obtain $Y' \succ_\tau^{<\alpha} d$, so still $X_2 \to_\gamma^\alpha d \equiv X_1$. We may thus take $X' \equiv X_1$.

If the contraction $X \to_s^\alpha X_1$ is not at the root, then assume without loss of generality that the contraction $X \to_w X_2$ is at the root. If $X \equiv \mathsf{K}X_2Y$ then $X_1 \equiv \mathsf{K}X_2'Y'$ with $X_2 \overset{\equiv}{\to}{}_s^\alpha X_2'$ and $Y \overset{\equiv}{\to}{}_s^\alpha Y'$. Hence $X_1 \to_w X_2'$ and we may take $X' \equiv X_2'$. So suppose $X \equiv \mathsf{S}Y_1Y_2Y_3$, $X_1 \equiv \mathsf{S}Y_1'Y_2'Y_3'$ and $X_2 \equiv Y_1Y_3(Y_2Y_3)$, where $Y_i \overset{\equiv}{\to}{}_s^\alpha Y_i'$. Then $X_2 \overset{*}{\to}{}_s^\alpha Y_1'Y_3'(Y_2'Y_3')$ and we may take $X' \equiv Y_1'Y_3'(Y_2'Y_3')$.

(b) Assume $X \succ_i^\alpha d$ and $X \to^\kappa X'$. We need to show $X' \succ_i^\alpha d$. We consider possible cases according to the definition of $X \succ_i^\alpha d$.

Assume $X \succ_i^\alpha d$ follows from ($\mathcal{F}_\tau$), i.e., $i = \tau = \mathsf{G}(\tau',F)$, $d \in \mathcal{D}_{\mathsf{G}(\tau',F)}$, and for every $a \in \mathcal{D}_{\tau'}$ we have $Xa \rightsquigarrow_{F(a)}^{<\alpha} d^{\mathcal{F}}(a)$. By ($\star$), for $a \in \mathcal{D}_{\tau'}$ we have $X'a \rightsquigarrow_{F(a)}^{<\alpha} d^{\mathcal{F}}(a)$. Thus $X' \succ_i^\alpha d$ by ($\mathcal{F}_\tau$).

Assume $X \succ_i^\alpha d$ follows from ($\mathcal{S}_\tau$), i.e., $i = \tau = \Upsilon(\tau',S)$, $X \succ_{\tau'}^{<\alpha} d$ and $d \in S$. By part (b) of the IH we obtain $X' \succ_{\tau'}^{<\alpha} d$. Thus $X' \succ_i^\alpha d$ by ($\mathcal{S}_\tau$).

Assume $X \succ_i^\alpha d$ follows from (π_τ^Σ), i.e., $i = \tau = \Sigma(\tau',F)$, $d = \langle a,b\rangle$, $a \in \mathcal{D}_{\tau'}$, $b \in \mathcal{D}_{F(a)}$, $\pi_1X \rightsquigarrow_{\tau'}^{<\alpha} a$ and $\pi_2X \rightsquigarrow_{F(a)}^{<\alpha} b$. Then by ($\star$) we obtain $\pi_1X' \rightsquigarrow_{\tau'}^{<\alpha} a$ and $\pi_2X' \rightsquigarrow_{F(a)}^{<\alpha} b$, and thus $X' \succ_i^\alpha d$ by (π_τ^Σ).

Assume $X \succ_i^\alpha d$ follows from $(\mathsf{G}_{\mathscr{T}})$, i.e., $i = \mathscr{T}$, $d = \mathsf{G}(\tau, F)$, $X \equiv \mathsf{G}AB$, $A \succ_{\mathscr{T}}^{<\alpha} \tau$ and for every $d \in \mathcal{D}_\tau$ we have $Bd \leadsto_{\mathscr{T}}^{<\alpha} F(d)$. Then $X' \equiv \mathsf{G}A'B'$ with $A \stackrel{\equiv}{\to}^{\kappa} A'$ and $B \stackrel{\equiv}{\to}^{\kappa} B'$. By part (b) of the IH we obtain $A' \succ_{\mathscr{T}}^{<\alpha} \tau$. By $(\star)$, for $d \in \mathcal{D}_\tau$ we have $B'd \leadsto_{\mathscr{T}}^{<\alpha} F(d)$. Hence $X' \equiv \mathsf{G}A'B' \succ_{\mathscr{T}}^{\alpha} \mathsf{G}(\tau, F)$ by $(\mathsf{G}_{\mathscr{T}})$.

Other cases are similar to the above or analogous to the corresponding cases in the proof of Lemma 6.2.6.

(c) Analogous to (b).

(d) Suppose $X \succ_i^\alpha d_1$ and $X \succ_i^\kappa d_2$. We need to show $d_1 = d_2$. We consider all possible overlaps of rules in Definition 7.2.3, i.e., all possible pairs of rules by which $X \succ_i^\alpha d_1$ and $X \succ_i^\kappa d_2$ could be obtained.

Assume both $X \succ_i^\alpha d_1$ and $X \succ_i^\kappa d_2$ follow from $(\mathcal{F}_\tau)$. Then $i = \tau = \mathsf{G}(\tau', F)$, $d_1, d_2 \in \mathcal{D}_\tau$ and for $a \in \mathcal{D}_{\tau'}$ we have $Xa \leadsto_{F(a)}^{<\alpha} d_1^{\mathcal{F}}(a)$ and $Xa \leadsto_{F(a)}^{<\kappa} d_2^{\mathcal{F}}(a)$. Then $d_1^{\mathcal{F}}(a) = d_2^{\mathcal{F}}(a)$ for $a \in \mathcal{D}_{\tau'}$, by $(\star\star)$. Thus $d_1 = d_2$.

Assume both $X \succ_i^\alpha d_1$ and $X \succ_i^\kappa d_2$ follow from $(\mathcal{S}_\tau)$. Then $i = \tau = \Upsilon(\tau', S)$, $d_1, d_2 \in S$, $X \succ_{\tau'}^{<\alpha} d_1$ and $X \succ_{\tau'}^{<\alpha} d_2$. By part (d) of the IH we obtain $d_1 \equiv d_2$.

Other cases are similar to the above or analogous to the corresponding cases in the proof of Lemma 6.2.6.

□

Like in Definition 6.2.7 we introduce the notion of the rank of a type. This notion is needed in the inductive proof of Lemma 7.2.9 – at certain points in the proof we need to show that the rank of the type considered decreases to be able to use the inductive hypothesis. Because here types are not finite objects, the rank of a type may be an infinite ordinal.

Definition 7.2.6. The *rank* of a type $\tau \in \mathscr{T}$, denoted $\mathrm{rank}(\tau)$, is an ordinal number defined inductively as follows.

$$\begin{array}{rcl} \mathrm{rank}(o) & = & 1 \\ \mathrm{rank}(\varepsilon) & = & 1 \\ \mathrm{rank}(\mathsf{G}(\tau, F)) & = & \max(\mathrm{rank}(\tau) + 1, \sup_{d \in \mathcal{D}_\tau} \mathrm{rank}(F(d))) \\ \mathrm{rank}(\Sigma(\tau, F)) & = & \max(\mathrm{rank}(\tau), \sup_{d \in \mathcal{D}_\tau} \mathrm{rank}(F(d))) \\ \mathrm{rank}(\mathsf{W}(\tau, F)) & = & \max(\mathrm{rank}(\tau), \sup_{d \in \mathcal{D}_\tau} (\mathrm{rank}(F(d)) + 1)) \\ \mathrm{rank}(\Upsilon(\tau, S)) & = & \mathrm{rank}(\tau) \end{array}$$

We write $X \gg^\kappa Y$ if there exists a term Z, distinct variables $x_1, \ldots, x_m \in \mathrm{FV}(X)$, and terms $X_1, \ldots, X_m$, $d_1, \ldots, d_m$ such that:

- $X \equiv Z[x_1/X_1, \ldots, x_m/X_m]$,
- $Y \equiv Z[x_1/d_1, \ldots, x_m/d_m]$,
- for each $k = 1, \ldots, m$ there is $\tau \in \mathscr{T}$ with $\mathrm{rank}(\tau) \leq \kappa$ and $X_k \succ_\tau d_k$.

We set $\gg^{<\kappa} = \bigcup_{\alpha<\kappa} \gg^\alpha$. We define a binary *subtype relation* $\sqsubseteq$ on $\mathscr{T}$ inductively:

- $\tau \sqsubseteq \tau$,
- if $\tau' \sqsubseteq \tau$ then $\Upsilon(\tau', S) \sqsubseteq \tau$.

Note that $\tau \sqsubseteq \tau'$ implies $\mathrm{rank}(\tau) = \mathrm{rank}(\tau')$.

Intuitively, one could equivalently think that when $X \succ_\tau d$, the rank is associated with the element d. The notion of rank used here bears some similarity to the notion of rank in ZFC set theory (see Section 2.2): the (types of) constituents (at least those that we care about) of an object have smaller rank than (the type of) the object itself. In fact, we could define, e.g.,

$$\mathrm{rank}(\mathsf{G}(\tau, F)) = \max(\mathrm{rank}(\tau), \sup_{d \in \mathcal{D}_\tau} \mathrm{rank}(F(d))) + 1$$

and the proofs would still go through. However, we need $\mathrm{rank}(\Upsilon(\tau, S)) = \mathrm{rank}(\tau)$, because $\mathcal{D}_{\Upsilon(\tau,S)} \subseteq \mathcal{D}_\tau$.

Lemma 7.2.7.

1. *If $X \succ_{\tau'} d$ and $\tau' \sqsubseteq \tau$ then $X \succ_\tau d$.*
2. *If $X \succ_\tau d$, $d \in \mathcal{D}_{\tau'}$ and $\tau' \sqsubseteq \tau$ then $X \succ_{\tau'} d$.*

Proof.

1. Induction on the definition of $\tau' \sqsubseteq \tau$. If $\tau = \tau'$ then the claim is obvious. Otherwise $\tau' = \Upsilon(\tau_0, S)$ with $\tau_0 \sqsubseteq \tau$. Then $X \succ_{\tau'} d$ could only be obtained by $(\mathcal{S}_{\tau'})$. So $X \succ_{\tau_0} d$, and by the inductive hypothesis $X \succ_\tau d$.
2. Induction on the definition of $\tau' \sqsubseteq \tau$. If $\tau = \tau'$ then the claim is obvious. Otherwise $\tau' = \Upsilon(\tau_0, S)$ with $\tau_0 \sqsubseteq \tau$. Then $d \in S \subseteq \mathcal{D}_{\tau_0}$, so by the inductive hypothesis $X \succ_{\tau_0} d$. Thus $X \succ_{\tau'} d$ by $(\mathcal{S}_{\tau'})$.

□

The following simple lemma will be used implicitly.

Lemma 7.2.8.

1. *If $X \gg^\kappa Y_1 Y_2$ then $X \equiv X_1 X_2$ with $X_1 \gg^\kappa Y_1$ and $X_2 \gg^\kappa Y_2$.*
2. *If $X \gg^\kappa NY$ where $N \in \{\mathsf{S}, \mathsf{K}, \Xi, \mathsf{\Lambda}, \mathsf{V}, \neg, \epsilon, \mathsf{M}, \mathsf{W}, \sup, \mathsf{T}, \mathsf{D}\}$ then $X \equiv NX'$ with $X' \gg^\kappa Y$.*
3. *If $X \gg^\kappa \mathsf{G} Y_1 Y_2$ then $X \equiv \mathsf{G} X_1 X_2$ with $X_i \gg^\kappa Y_i$. An analogous result holds when $X \gg^\kappa \Sigma Y_1 Y_2$ or $X \gg^\kappa \Upsilon Y_1 Y_2$.*

Proof. Follows directly from Definition 7.2.6. □

Lemma 7.2.9. *The reduction system R is invariant.*

Proof. Like in Lemma 6.2.9, we show the following two conditions by induction on pairs $\langle \kappa, \alpha \rangle$ ordered lexicographically, i.e., $\langle \kappa_1, \alpha_1 \rangle < \langle \kappa_2, \alpha_2 \rangle$ iff $\kappa_1 < \kappa_2$, or $\kappa_1 = \kappa_2$ and $\alpha_1 < \alpha_2$.

(1) If $X \gg^\kappa Y \succ_i^\alpha d$ then $X \succ_i d$, where $i \in \mathscr{T}$ or $i = \mathscr{T}$.

(2) If $X \gg^\kappa Y \xrightarrow{*}^\alpha Z$ then there is Y' with $X \xrightarrow{*} Y' \gg^\kappa Z$.

For $\kappa = \alpha = \zeta$, where ζ is the closure ordinal of Definition 7.2.3, the above conditions imply the invariance of R. Indeed, assume $X \succ_i d$ and $Yd \rightsquigarrow_j d'$. Then $YX \gg^\kappa Yd \xrightarrow{*}^\kappa \cdot \succ_j^\kappa d'$ for some κ. Using (2) and then (1) we obtain $YX \rightsquigarrow_j d'$.

So assume (1) and (2) hold for all $\langle \kappa', \alpha' \rangle < \langle \kappa, \alpha \rangle$. As in the proof of Lemma 6.2.9, one may show the following condition.

($\star$) If $X \gg^\kappa Y \rightsquigarrow_\tau^{<\alpha} d$ then $X \rightsquigarrow_\tau d$.

Now we show (1) and (2) for $\langle \kappa, \alpha \rangle$.

(1) Assume $X \gg^\kappa Y \succ_i^\alpha d$ where $i \in \mathscr{T}$ or $i = \mathscr{T}$. We consider all possible rules by which $Y \succ_i^\alpha d$ could be obtained.

- ($\mathcal{D}_\tau$) Then $X \gg^\kappa d \succ_\tau^\alpha d$ with $\tau = o$ or $\tau = \mathsf{W}(\tau_0, F)$. This is only possible when $X \equiv d$ or $X \succ_{\tau'} d$. If $X \succ_{\tau'} d$ then $\tau' \sqsubseteq \tau$. By Lemma 7.2.7 we obtain $X \succ_\tau d$.
- ($\mathcal{F}_\tau$) Then $X \gg^\kappa Y \succ_\tau^\alpha d$, $\tau = \mathsf{G}(\tau', F)$, $d \in \mathcal{D}_\tau$, and for every $a \in \mathcal{D}_{\tau'}$ we have $Ya \rightsquigarrow_{F(a)}^{<\alpha} d^{\mathcal{F}}(a)$. Let $a \in \mathcal{D}_{\tau'}$. Then $Xa \gg^\kappa Ya \rightsquigarrow_{F(a)}^{<\alpha} d^{\mathcal{F}}(a)$. Thus $Xa \rightsquigarrow_{F(a)} d^{\mathcal{F}}(a)$ by ($\star$). Since $a \in \mathcal{D}_{\tau'}$ was arbitrary, we conclude $X \succ_\tau d$.
- ($\mathsf{T}_\top$) Then $X \gg^\kappa \mathsf{T}\langle d, f \rangle^\tau Y_1 \succ_o^\alpha \top$, $\tau = \mathsf{W}(\tau', F)$ and $Y_1 \succ_{\tau'}^{<\alpha} d$. Because $\tau = \mathsf{W}(\tau', F)$, we have $X \equiv \mathsf{T}\langle d, f \rangle^\tau Y_0$ with $Y_0 \gg^\kappa Y_1$. By part (1) of the IH we obtain $Y_0 \succ_{\tau'} d$. Thus $X \succ_o \top$.
- (π_τ^Σ) Then $X \gg^\kappa Y \succ_\tau^\alpha \langle a, b \rangle$, $\tau = \Sigma(\tau', F)$, $a \in \mathcal{D}_{\tau'}$, $b \in \mathcal{D}_{F(a)}$, $\pi_1 Y \rightsquigarrow_{\tau'}^{<\alpha} a$ and $\pi_2 Y \rightsquigarrow_{F(a)}^{<\alpha} b$. We have $\pi_1 X \gg^\kappa \pi_1 Y$ and $\pi_2 X \gg^\kappa \pi_2 Y$, so $\pi_1 X \rightsquigarrow_{\tau'} a$ and $\pi_2 X \rightsquigarrow_{F(a)} b$ by ($\star$).
- ($\mathsf{G}_\mathscr{T}$) Then $X \gg^\kappa \mathsf{G} Y_1 Y_2 \succ_\mathscr{T}^\alpha \mathsf{G}(\tau, F)$, $Y_1 \succ_\mathscr{T}^{<\alpha} \tau$ and for every $d \in \mathcal{D}_\tau$ we have $Y_2 d \rightsquigarrow_\mathscr{T}^{<\alpha} F(d)$. We have $X \equiv \mathsf{G} X_1 X_2$ with $X_1 \gg^\kappa Y_1$ and $X_2 \gg^\kappa Y_2$. Thus $X_1 \gg^\kappa Y_1 \succ_\mathscr{T}^{<\alpha} \tau$, so by part (1) of the IH we obtain $X_1 \succ_\mathscr{T} \tau$. Let $d \in \mathcal{D}_\tau$. We have $X_2 d \gg^\kappa Y_2 d \rightsquigarrow_\mathscr{T}^{<\alpha} F(d)$, so $X_2 d \rightsquigarrow_\mathscr{T} F(d)$ by ($\star$). Thus $X \succ_\mathscr{T} \mathsf{G}(\tau, F)$.

Other cases are similar to the above or analogous to corresponding cases in the proof of Lemma 6.2.9.

(2) It suffices to show that if $X \gg^\kappa Y \to^\alpha Z$ then $X \xrightarrow{*} \cdot \gg^\kappa Z$. Without loss of generality, we may assume that the contraction $Y \to^\alpha Z$ occurs at the root. We consider possible rules by which this contraction could occur.

- (w_1) We have $X \gg^\kappa \mathsf{K} Y_1 Y_2 \to_w Y_1$. Then $X \equiv \mathsf{K} X_1 X_2$ with $X_i \gg^\kappa Y_i$. Hence $X \to_w X_1 \gg^\kappa Y_1$.
- (w_2) We have $X \gg^\kappa \mathsf{S} Y_1 Y_2 Y_3 \to_w Y_1 Y_3 (Y_2 Y_3)$. Then $X \equiv \mathsf{S} X_1 X_2 X_3$ with $X_i \gg^\kappa Y_i$. Hence $X \to_w X_1 X_3 (X_2 X_3) \gg^\kappa Y_1 Y_3 (Y_2 Y_3)$.
- (γ) We have $X \gg^\kappa Y \to_\gamma^\alpha Z$. There are two possibilities.

1. $X \equiv fX'$, $Y \equiv fY'$, $X' \gg^\kappa Y'$ and $f \in \mathcal{D}_{\mathsf{G}(\tau,F)}$. Then $Y' \succ_\tau^{<\alpha} d$ for some $d \in \mathcal{D}_\tau$ and $Z \equiv f^{\mathcal{F}}(d)$. By part (1) of the IH we have $X' \succ_\tau d$. So $X \equiv fX' \to_\gamma f^{\mathcal{F}}(d) \equiv Z \gg^\kappa Z$.
2. $X \equiv MX'$, $Y \equiv fY'$, $M \succ_\tau f$, $X' \gg^\kappa Y'$, $\mathrm{rank}(\tau) \leq \kappa$ and $\tau \sqsubseteq \mathsf{G}(\tau_1, F)$. Then $Y' \succ_{\tau_1}^{<\alpha} d$ for some $d \in \mathcal{D}_{\tau_1}$ and $Z \equiv f^{\mathcal{F}}(d)$. Since $X' \gg^\kappa Y' \succ_{\tau_1}^{<\alpha} d$, we have $X' \succ_{\tau_1} d$ by part (1) of the IH. Let $\tau_0 = \mathsf{G}(\tau_1, F)$. Since $\tau \sqsubseteq \tau_0$, by Lemma 7.2.7 we also have $M \succ_{\tau_0} f$. One sees by inspecting Definition 7.2.3 that $M \succ_{\tau_0} f$ can only be obtained by $(\mathcal{F}_{\tau_0})$. Since $d \in \mathcal{D}_{\tau_1}$ we thus have $Md \rightsquigarrow_{F(d)} f^{\mathcal{F}}(d) \equiv Z$. Because $\mathrm{rank}(\tau_1) < \mathrm{rank}(\tau_0) \leq \kappa$ and $X' \succ_{\tau_1} d$, we have $MX' \gg^{<\kappa} Md \xrightarrow{*} \cdot \succ_{F(d)} Z$. Thus $MX' \xrightarrow{*} \cdot \gg^{<\kappa} \cdot \succ_{F(d)} Z$ by part (2) of the IH. So $MX' \xrightarrow{*} \cdot \succ_{F(d)} Z$ by part (1) of the IH. Since $\mathrm{rank}(F(d)) \leq \mathrm{rank}(\tau_0) \leq \kappa$ we have $X \equiv MX' \xrightarrow{*} \cdot \gg^\kappa Z$.

(ϵ_1) We have $X \gg^\kappa \epsilon A Y_1 \to_\epsilon^\alpha d$ with $A \succ_{\mathscr{T}}^{<\alpha} \tau$, $Y_1 \succ_{\tau \to o}^{<\alpha} p$ and $d = \eta_\tau(\mathcal{D}_{\tau,p})$. We need to find Y' with $X \xrightarrow{*} Y' \gg^\kappa d$. We have $X \equiv \epsilon A_0 Y_0$ with $A_0 \gg^\kappa A$ and $Y_0 \gg^\kappa Y_1$. It suffices to show that $A_0 \succ_{\mathscr{T}} \tau$ and $Y_0 \succ_{\tau \to o} p$. We have $A_0 \gg^n A \succ_{\mathscr{T}}^{<\alpha} \tau$, so $A_0 \succ_{\mathscr{T}} \tau$ by part (1) of the IH. Also $Y_0 \gg^\kappa Y_1 \succ_{\tau \to o}^{<\alpha} p$, so $Y_0 \succ_{\tau \to o} p$ by part (1) of the IH.

(ϵ_2) Analogous to the case for (ϵ_1).

(μ_1) We have $X \gg^\kappa \mathsf{M} Y_1 \to_\mu^\alpha \mathsf{K}$ and $Y_1 \succ_o^{<\alpha} \top$. Then $X \equiv \mathsf{M} X_1$ with $X_1 \gg^\kappa Y_1$. So $X_1 \succ_o \top$ by part (1) of the IH. Thus $X \to_\mu \mathsf{K}$.

(μ_2) Analogous to the case for (μ_1).

(π_1) We have $X \gg^\kappa \pi_1 \langle a, b \rangle^\tau \to_\pi a$ with $\tau = \Sigma(\tau', F)$, and either $X \equiv Y$, which case is trivial, or $X \equiv \pi_1 X'$ with $X' \succ_{\tau_1} \langle a, b \rangle$, $\mathrm{rank}(\tau_1) \leq \kappa$. Then $X' \succ_{\tau_1} \langle a, b \rangle$ must follow by rule (π_τ^Σ) (and possibly some applications of ($\mathcal{S}_\tau$) but these may be ignored by Lemma 7.2.7), so we have $\tau_1 = \tau$. Then $a \in \mathcal{D}_{\tau'}$ and $\pi_1 X' \rightsquigarrow_{\tau'} a$, i.e., $X \equiv \pi_1 X' \xrightarrow{*} Y' \succ_{\tau'} a$. Since $\mathrm{rank}(\tau') \leq \mathrm{rank}(\tau) \leq \kappa$, we actually have $Y' \gg^\kappa a$.

(π_2) We have $X \gg^\kappa \pi_2 \langle d, f \rangle^\tau \to f$ with $\tau = \Sigma(\tau', F)$. The argument is analogous to the case for (π_1).

(sup) We have $X \gg^\kappa \sup A Y_1 Y_2 \to^\alpha \langle d, f \rangle^\tau$ where $A \succ_{\mathscr{T}}^{<\alpha} \tau = \mathsf{W}(\tau', F)$, $Y_1 \succ_{\tau'}^{<\alpha} d$ and $Y_2 \succ_{F(d) \to \tau}^{<\alpha} f$. Then $X \equiv \sup A' X_1 X_2$, $A' \gg^\kappa A$, $X_1 \gg^\kappa Y_1$ and $X_2 \gg^\kappa Y_2$. By part (1) of the IH we obtain $A' \succ_{\mathscr{T}} \tau$, $X_1 \succ_{\tau'} d$ and $X_2 \succ_{F(d) \to \tau'} f$. Thus $X \to \langle d, f \rangle^\tau$.

(D) We have $X \gg^\kappa \mathsf{D} \langle d, f \rangle^\tau Y_1 \to^\alpha f Y_1$ with $\tau = \mathsf{W}(\tau', F)$. Then $X \equiv \mathsf{D} \langle d, f \rangle^\tau X_1$ with $X_1 \gg^\kappa Y_1$. So $X \to f X_1 \gg f Y_1$.

□

We have thus established coherence and invariance of the system R. Like in Section 6.2, it remains to show some lemmas corresponding to the conditions 1-29 in Definition 7.1.8. The proofs of these lemmas are mostly straightforward, using coherence and invariance of the system R.

Definition 7.2.10. We define the *size* of a type τ, denoted $|\tau|$, to be the ordinal number given by the following inductive definition:

- $|\varepsilon| = |o| = 1$,
- $|\Upsilon(\tau, S)| = |\tau| + 1$,
- $|\mathsf{G}(\tau, F)| = \max(|\tau|, \sup_{d \in \mathcal{D}_\tau} |F(d)|) + 1$,
- $|\Sigma(\tau, F)| = \max(|\tau|, \sup_{d \in \mathcal{D}_\tau} |F(d)|) + 1$,
- $|\mathsf{W}(\tau, F)| = \max(|\tau|, \sup_{d \in \mathcal{D}_\tau} |F(d)|) + 1$.

Lemma 7.2.11. *If $d \in \mathcal{D}_{\tau'}$ for $\tau' \sqsubseteq \tau \in \mathscr{T}$, then $d^\tau \succ_{\tau'} d^\tau$.*

Proof. Induction on the size of τ. ☐

The above lemma implies that if $X \xrightarrow{*} d$ for $d \in \mathcal{D}_\tau$, then $X \rightsquigarrow_\tau d$. We will sometimes use this property implicitly.

Lemma 7.2.12. *If $X \succ_{\mathscr{T}} \tau$ then for any Z with $XZ \rightsquigarrow_o \top$ there is $d \in \mathcal{D}_\tau$ with $Z \rightsquigarrow_\tau d$.*

Proof. By induction on pairs $\langle |\tau|, \alpha \rangle$ ordered lexicographically we show that if $X \succ_{\mathscr{T}} \tau$ and $XZ \rightsquigarrow_o^\alpha \top$ then there exists $d \in \mathcal{D}_\tau$ such that $Z \rightsquigarrow_\tau d$. Suppose $X \succ_{\mathscr{T}} \tau$ and $XZ \rightsquigarrow_o^\alpha \top$.

If $\tau = o$ then $X \equiv \mathsf{H}$, and $\mathsf{H}Z \rightsquigarrow_o \top$. By coherence we have $Z \vee \neg Z \rightsquigarrow_o \top$. This implies $Z \rightsquigarrow_o \top$ or $Z \rightsquigarrow_o \bot$, and we are done because $\top, \bot \in \mathcal{D}_o$. If $X \succ_{\mathscr{T}} \varepsilon$ then $X \equiv \mathsf{O}$ and $\mathsf{O}Z \xrightarrow{*} \bot$, so $XZ \rightsquigarrow_o \top$ is impossible by coherence.

Assume $X \succ_{\mathscr{T}} \mathsf{G}(\tau, F)$ follows by $(\mathsf{G}_{\mathscr{T}})$, and $XZ \rightsquigarrow_o \top$. Then $X \equiv \mathsf{G}X_1X_2$ with $X_1 \succ_{\mathscr{T}} \tau$ and for every $d \in \mathcal{D}_\tau$ we have $X_2 d \rightsquigarrow_{\mathscr{T}} F(d)$. Since $XZ \rightsquigarrow_o \top$, we have $\mathsf{G}X_1X_2Z \rightsquigarrow_o \top$. Let $d \in \mathcal{D}_\tau$. By coherence and $(\Xi_\top)$ we have $X_2 d(Zd) \rightsquigarrow_o \top$. Since $X_2 d \rightsquigarrow_{\mathscr{T}} F(d)$ and $|F(d)| < |\mathsf{G}(\tau, F)|$, by the IH and coherence there is $a_d \in \mathcal{D}_{F(d)}$ with $Zd \rightsquigarrow_{F(d)} a_d$. So by $(\mathcal{F}_{\mathsf{G}(\tau,F)})$ we have $Z \succ_{\mathsf{G}(\tau,F)} f$ for $f \in \mathcal{D}_{\mathsf{G}(\tau,F)}$ such that $f^{\mathcal{F}}(d) = a_d$ for $d \in \mathcal{D}_\tau$.

Assume $X \succ_{\mathscr{T}}^\alpha \Sigma(\tau, F)$ and $XZ \rightsquigarrow_o \top$. Then $X \equiv \Sigma AB$ where $A \succ_{\mathscr{T}} \tau$ and for each $d \in \mathcal{D}_\tau$ we have $Bd \rightsquigarrow_{\mathscr{T}} F(d)$. Since $XZ \rightsquigarrow_o \top$, we have $A(\pi_1 Z) \rightsquigarrow_o \top$ and $B(\pi_1 Z)(\pi_2 Z) \rightsquigarrow_o \top$. Since $|\tau| < |\Sigma(\tau, F)|$, by the IH there is $d \in \mathcal{D}_\tau$ with $\pi_1 Z \rightsquigarrow_\tau d$. Since $Bd \rightsquigarrow_{\mathscr{T}} F(d)$ and $\pi_1 Z \rightsquigarrow_\tau d$, we have $B(\pi_1 Z) \rightsquigarrow_{\mathscr{T}} F(d)$ by invariance. Because $B(\pi_1 Z) \rightsquigarrow_{\mathscr{T}} F(d)$, $B(\pi_1 Z)(\pi_2 Z) \rightsquigarrow_o \top$ and $|F(d)| < |\Sigma(\tau, F)|$, by the IH there is $d' \in \mathcal{D}_{F(d)}$ with $\pi_2 Z \rightsquigarrow_{F(d)} d'$. Then $Z \rightsquigarrow_{\Sigma(\tau,F)} \langle d, d' \rangle$ by $(\pi^\Sigma_{\Sigma(\tau,F)})$.

Assume $X \succ_{\mathscr{T}} \tau = \mathsf{W}(\tau', F)$ and $XZ \rightsquigarrow_o^\alpha \top$. Then $X \equiv \mathsf{W}AB$. Also $XZ \equiv \mathsf{W}ABZ \xrightarrow{*} \mathsf{W}A'B'Z' \succ_o^\alpha \top$, which implies $Z' \equiv \langle d_0, f \rangle^{\tau_1}$ and $\mathsf{W}A'B' \succ_{\mathscr{T}}^{<\alpha} \tau_1$. Because $A \xrightarrow{*} A'$ and $B \xrightarrow{*} B'$, we have $X \equiv \mathsf{W}AB \xrightarrow{*} \mathsf{W}A'B'$, and by coherence we conclude that $\mathsf{W}A'B' \succ_{\mathscr{T}} \tau$. Therefore $Z \rightsquigarrow_\tau \langle d_0, f \rangle \in \mathcal{D}_\tau$.

So assume $X \succ_{\mathscr{T}} \tau = \Upsilon(\tau', S_p)$, $XZ \rightsquigarrow_o \top$ and $X \equiv \Upsilon AY$ with $A \succ_{\mathscr{T}} \tau'$, $Y \succ_{\tau' \to o} p$ and $S_p = \{d \in \mathcal{D}_{\tau'} \mid p^{\mathcal{F}}(d) \equiv \top\}$. Hence by coherence, by $XZ \rightsquigarrow_o \top$ and by $(\wedge_\top)$ we have $AZ \rightsquigarrow_o \top$ and $YZ \rightsquigarrow_o \top$. Since $|\tau'| < |\tau|$, by the IH there is $d \in \mathcal{D}_{\tau'}$ with $Z \rightsquigarrow_{\tau'} d$. If $p^{\mathcal{F}}(d) \equiv \top$ then $d \in \mathcal{D}_\tau$ and $Z \rightsquigarrow_\tau d$ by $(\mathcal{S}_\tau)$. Otherwise we have $pd \rightarrow_\gamma p^{\mathcal{F}}(d) \equiv \bot$ because $d \succ_{\tau'} d$ by Lemma 7.2.11. So $Yd \rightsquigarrow_o \bot$ by coherence and invariance, because $Y \succ_{\tau' \to o} p$. Since $Z \rightsquigarrow_{\tau'} d$ we obtain $YZ \rightsquigarrow_o \bot$ by coherence and invariance. Thus $XZ = AZ \wedge YZ \rightsquigarrow_o \bot$. But since $XZ \rightsquigarrow_o \top$ this contradicts coherence. ☐

Lemma 7.2.13. *If $X \succ_{\mathscr{T}} \tau$ then $Xd \leadsto_o \top$ for any $d \in \mathcal{D}_\tau$.*

Proof. Induction on the size of τ. Suppose $X \succ_{\mathscr{T}} \tau$ and $d \in \mathcal{D}_\tau$.

If $\tau = o$ then $X \equiv \mathsf{H}$, $d \in \{\top, \bot\}$, and $\mathsf{H}d \succ_o \top$ follows from definitions. The case $\tau = \varepsilon$ is trivial because $\mathcal{D}_\varepsilon = \emptyset$.

Assume $X \succ_{\mathscr{T}} \mathsf{G}(\tau, F)$ follows by $(\mathsf{G}_{\mathscr{T}})$, and $d \in \mathcal{D}_{\mathsf{G}(\tau,F)}$. Then $X \equiv \mathsf{G}X_1X_2$ where $X_1 \succ_{\mathscr{T}} \tau$ and for every $a \in \mathcal{D}_\tau$ we have $X_2a \leadsto_{\mathscr{T}} F(a)$. Let $a \in \mathcal{D}_\tau$. Then $X_2a(d^{\mathcal{F}}(a)) \leadsto_o \top$ by the IH. Since $a \succ_\tau a$ by Lemma 7.2.11, we have $da \to_\gamma d^{\mathcal{F}}(a)$, and thus $X_2a(da) \leadsto_o \top$. Hence $Xd \leadsto_o \top$ by $(\Xi_\top)$ and coherence.

Assume $X \succ_{\mathscr{T}}^{\alpha} \tau = \Sigma(\tau', F)$ and $d \in \mathcal{D}_\tau$. Then $X \equiv \Sigma AB$ and $d = \langle d_1, d_2 \rangle$ with $d_1 \in \mathcal{D}_{\tau'}$ and $d_2 \in \mathcal{D}_{F(d_1)}$. We have $Xd = A(\pi_1 d) \wedge B(\pi_1 d)(\pi_2 d) = Ad_1 \wedge Bd_1d_2$. By $X \succ_{\mathscr{T}} \tau$ and $d_1 \in \mathcal{D}_{\tau'}$ we have: $A \succ_{\mathscr{T}} \tau'$ and $Bd_1 \leadsto_{\mathscr{T}}^{\leq\alpha} F(d_1)$. Hence by the IH and coherence, $Ad_1 \leadsto_o \top$ and $Bd_1d_2 \leadsto_o \top$. Thus $Xd \leadsto_o \top$.

Assume $X \succ_{\mathscr{T}} \tau = \mathsf{W}(\tau', F)$. Then $X \equiv \mathsf{W}AB$ and $d = \langle d_0, f \rangle \in \mathcal{D}_\tau$. By $(\mathsf{W}_\top)$ we obtain $Xd \succ_o \top$.

So assume $X \succ_{\mathscr{T}} \tau = \Upsilon(\tau', S_p)$, $d \in \mathcal{D}_\tau$ and $X \equiv \Upsilon AY$ with $A \succ_{\mathscr{T}} \tau'$, $Y \succ_{\tau' \to o} p$ and $S_p = \{d \in \mathcal{D}_{\tau'} \mid p^{\mathcal{F}}(d) \equiv \top\}$. We have $d \in \mathcal{D}_\tau = S_p$, so $d \in \mathcal{D}_{\tau'}$ and $p^{\mathcal{F}}(d) \equiv \top$. By the inductive hypothesis $Ad \leadsto_o \top$. Since $d \succ_{\tau'} d$ by Lemma 7.2.11 we have $pd \to_\gamma p^{\mathcal{F}}(d) \equiv \top$, i.e., $pd \leadsto_o \top$. Thus $Yd \leadsto_o \top$ by coherence and invariance. Hence $Ad \wedge Yd \leadsto_o \top$, so $Xd \leadsto_o \top$ by coherence. □

Lemma 7.2.14. *The following conditions hold.*

1. *$\Xi XY \leadsto_o \top$ iff $\mathsf{L}X \leadsto_o \top$ and for every Z with $XZ \leadsto_o \top$ we have $YZ \leadsto_o \top$.*
2. *$\Xi XY \leadsto_o \bot$ iff $\mathsf{L}X \leadsto_o \top$ and there exists Z with $XZ \leadsto_o \top$ and $YZ \leadsto_o \bot$.*

Proof. Follows from Lemma 7.2.5, Lemma 7.2.9, Lemma 7.2.12 and Lemma 7.2.13. □

Lemma 7.2.15. *If $p \in \mathcal{D}_{\tau_1 \to \tau_2}$ and $pX \leadsto_{\tau_2} b$ for some $b \in \mathcal{D}_{\tau_2}$, then there is $a \in \mathcal{D}_{\tau_1}$ with $X \leadsto_{\tau_1} a$ and $p^{\mathcal{F}}(a) \equiv b$.*

Proof. The proof is completely analogous to the proof of Lemma 6.2.16. In the inductive proof of $(\star)$ one needs to consider additional cases according to Definition 7.2.3. □

Lemma 7.2.16. *If $\mathsf{Q}_L AXY \leadsto_o \top$ and $A \leadsto_{\mathscr{T}} \tau$ then there is $d \in \mathcal{D}_\tau$ such that $X \leadsto_\tau d$ and $Y \leadsto_\tau d$.*

Proof. The proof is completely analogous to the proof of Lemma 6.2.17, but instead of Lemma 6.2.16 we use Lemma 7.2.15. □

Lemma 7.2.17. *If $\mathsf{Q}_L AXY \leadsto_o \bot$ and $A \leadsto_{\mathscr{T}} \tau$ then there are $d_1, d_2 \in \mathcal{D}_\tau$ such that $d_1 \neq d_2$, $X \leadsto_\tau d_1$ and $Y \leadsto_\tau d_2$.*

Proof. Recall that $\mathsf{Q}_L AXY = \Xi(\mathsf{F}A\mathsf{H})(\lambda p.\neg(pX) \vee pY)$. Assume $\mathsf{Q}_L AXY \leadsto_o \bot$ and $A \leadsto_{\mathscr{T}} \tau$. Then $\mathsf{F}A\mathsf{H} \leadsto_{\mathscr{T}} \tau \to o$. Because $\mathsf{Q}_L AXY \leadsto_o \bot$, there is $p \in \mathcal{D}_{\tau \to o}$ such that $\neg(pX) \vee pY \leadsto_o \bot$, i.e., $pX \leadsto_o \top$ and $pY \leadsto_o \bot$. Hence by Lemma 7.2.15 there are $d_1, d_2 \in \mathcal{D}_\tau$ such that $p^{\mathcal{F}}(d_1) \equiv \top$, $p^{\mathcal{F}}(d_2) \equiv \bot$, $X \leadsto_\tau d_1$ and $Y \leadsto_\tau d_2$. Then also $d_1 \neq d_2$. □

Lemma 7.2.18. *If* $\mathsf{W}AB(\sup(\mathsf{W}AB)XY) \rightsquigarrow_o \top$ *then* $\mathsf{W}AB \rightsquigarrow_{\mathscr{T}} \mathsf{W}(\tau, F)$, $X \rightsquigarrow_\tau d$ *for some* $d \in \mathcal{D}_\tau$, *and* $Y \rightsquigarrow_{F(d)\to\mathsf{W}(\tau,F)} f$ *for some* $f \in \mathcal{D}_{F(d)\to\mathsf{W}(\tau,F)}$.

Proof. Note that in order to obtain $\mathsf{W}AB(\sup(\mathsf{W}AB)XY) \rightsquigarrow_o \top$ one must use the rules $(\mathsf{W}_\top)$ and (sup). □

Lemma 7.2.19.

1. *If* $\mathsf{W}AB(\sup(\mathsf{W}AB)XY) \rightsquigarrow_o \top$ *and* $\mathsf{Q}_L AXX' \rightsquigarrow_o \top$ *then* $\mathsf{T}(\sup(\mathsf{W}AB)XY)X' \rightsquigarrow_o \top$.
2. *If* $\mathsf{W}AB(\sup(\mathsf{W}AB)XY) \rightsquigarrow_o \top$ *and* $\mathsf{Q}_L AXX' \rightsquigarrow_o \bot$ *then* $\mathsf{T}(\sup(\mathsf{W}AB)XY)X' \rightsquigarrow_o \bot$.
3. *If* $\mathsf{W}AB(\sup(\mathsf{W}AB)XY) \rightsquigarrow_o \top$ *and* $BXZ \rightsquigarrow_o \top$ *then* $\mathsf{D}(\sup(\mathsf{W}AB)XY)Z =_R YZ$.

Proof.

1. Assume $\mathsf{W}AB(\sup(\mathsf{W}AB)XY) \rightsquigarrow_o \top$ and $\mathsf{Q}_L AXX' \rightsquigarrow_o \top$. By Lemma 7.2.18 we have $\mathsf{W}AB \rightsquigarrow_{\mathscr{T}} \mathsf{W}(\tau, F)$, $X \rightsquigarrow_\tau d$ for some $d \in \mathcal{D}_\tau$ and $Y \rightsquigarrow_{F(d)\to\mathsf{W}(\tau,F)} f$ for some $f \in \mathcal{D}_{F(d)\to\mathsf{W}(\tau,F)}$. Thus $\sup(\mathsf{W}AB)XY \stackrel{*}{\to} \langle d, f\rangle$. By Lemma 7.2.16 there is $d' \in \mathcal{D}_\tau$ such that $X \rightsquigarrow_\tau d'$. By coherence $d \equiv d'$. Hence $\mathsf{T}(\sup(\mathsf{W}AB)XY)X' \rightsquigarrow_o \top$ by $(\mathsf{T}_\top)$.
2. Analogous to the previous point, using Lemma 7.2.17 and $(\mathsf{T}_\bot)$.
3. Assume $\mathsf{W}AB(\sup(\mathsf{W}AB)XY) \rightsquigarrow_o \top$ and $BXZ \rightsquigarrow_o \top$. By Lemma 7.2.18 we have $\mathsf{W}AB \rightsquigarrow_{\mathscr{T}} \mathsf{W}(\tau, F)$, $X \rightsquigarrow_\tau d$ for some $d \in \mathcal{D}_\tau$ and $Y \rightsquigarrow_{F(d)\to\mathsf{W}(\tau,F)} f$ for some $f \in \mathcal{D}_{F(d)\to\mathsf{W}(\tau,F)}$. Thus $\sup(\mathsf{W}AB)XY \stackrel{*}{\to} \langle d, f\rangle$. Then $\mathsf{D}(\sup(\mathsf{W}AB)XY)Z \stackrel{*}{\to} fZ$. We have $Bd \rightsquigarrow_{\mathscr{T}} F(d)$. Thus $BX \rightsquigarrow_{\mathscr{T}} F(d)$ by invariance. By Lemma 7.2.12 there is $b \in \mathcal{D}_{F(d)}$ with $Z \rightsquigarrow_{F(d)} b$. Then $fZ \to_\gamma f^{\mathcal{F}}(b)$. Since $Y \rightsquigarrow_{F(d)\to\mathsf{W}(\tau,F)} f$, by invariance $YZ \rightsquigarrow_{\mathsf{W}(\tau,F)} f^{\mathcal{F}}(b)$, i.e., $YZ \stackrel{*}{\to} Y' \succ_{\mathsf{W}(\tau,F)} f^{\mathcal{F}}(b)$. But $Y' \succ_{\mathsf{W}(\tau,F)} f^{\mathcal{F}}(b)$ is only possible when $Y' \equiv f^{\mathcal{F}}(b)$. Thus $YZ \stackrel{*}{\to} f^{\mathcal{F}}(b)$. Also $\mathsf{D}(\sup(\mathsf{W}AB)XY)Z \stackrel{*}{\to} fZ \to f^{\mathcal{F}}(b)$. Therefore $\mathsf{D}(\sup(\mathsf{W}AB)XY)Z = YZ$.

□

Lemma 7.2.20.

1. *If* $\mathsf{X}AY \rightsquigarrow_o \top$ *and* $\mathsf{F}A\mathsf{H}Y \rightsquigarrow_o \top$ *then* $Y(\epsilon AY) \rightsquigarrow_o \top$.
2. *If* $\mathsf{X}AA \rightsquigarrow_o \top$ *and* $\mathsf{F}A\mathsf{H}Y \rightsquigarrow_o \top$ *then* $A(\epsilon AY) \rightsquigarrow_o \top$.

Proof. Recall that $\mathsf{X}AY = \neg(\Xi A(\lambda x.\neg(Yx)))$.

1. Assume $\mathsf{X}AY \rightsquigarrow_o \top$ and $\mathsf{F}A\mathsf{H}Y \rightsquigarrow_o \top$. Then $\Xi A(\lambda x.\neg(Yx)) \rightsquigarrow_o \bot$, so $A \rightsquigarrow_{\mathscr{T}} \tau$ for some $\tau \in \mathscr{T}$, and there is $d \in \mathcal{D}_\tau$ such that $\neg(Yd) \rightsquigarrow_o \bot$, i.e., $Yd \rightsquigarrow_o \top$. We have $\mathsf{F}A\mathsf{H} \rightsquigarrow_{\mathscr{T}} \tau \to o$. So by Lemma 7.2.12 there is $p \in \mathcal{D}_{\tau\to o}$ with $Y \rightsquigarrow_{\tau\to o} p$. Because $p^{\mathcal{F}}(d) = \top$, we have $\mathcal{D}_{\tau,p} \neq \emptyset$. Thus $\epsilon AY \to \eta_\tau(\mathcal{D}_{\tau,p})$. By the definition of $\mathcal{D}_{\tau,p}$ we have $p^{\mathcal{F}}(\eta_\tau(\mathcal{D}_{\tau,p})) = \top$. So $p(\eta_\tau(\mathcal{D}_{\tau,p})) \to_\gamma \top$, using Lemma 7.2.11. By Lemma 7.2.11, coherence and invariance we obtain $Y(\eta_\tau(\mathcal{D}_{\tau,p})) \rightsquigarrow_o \top$. Therefore $Y(\epsilon AY) \rightsquigarrow_o \top$.
2. Assume $\mathsf{X}AA \rightsquigarrow_o \top$ and $\mathsf{F}A\mathsf{H}Y \rightsquigarrow_o \top$. Then $A \rightsquigarrow_{\mathscr{T}} \tau$ for some $\tau \in \mathscr{T}$, and $\mathcal{D}_\tau \neq \emptyset$. As in the previous paragraph, there is $p \in \mathcal{D}_{\tau\to o}$ with $Y \rightsquigarrow_{\tau\to o} p$. We thus have $\epsilon AY \to_\epsilon d$ for some $d \in \mathcal{D}_\tau$. By Lemma 7.2.13 we obtain $Ad \rightsquigarrow_o \top$. Hence $A(\epsilon AY) \rightsquigarrow_o \top$.

□

Lemma 7.2.21. *If the following conditions hold*

1. $AX \rightsquigarrow_o \top$,
2. $\mathsf{F}(BX)(\mathsf{W}AB)Y \rightsquigarrow_o \top$,
3. $\mathsf{L}(\mathsf{W}AB) \rightsquigarrow_o \top$,

then $\mathsf{W}AB(\sup(\mathsf{W}AB)XY) \rightsquigarrow_o \top$.

Proof. Follows from (sup), $(\mathsf{W}_\top)$, Lemma 7.2.12, invariance and definitions. □

Lemma 7.2.22. *If* $\mathsf{L}X \rightsquigarrow_o \top$ *and for every* Z *such that* $XZ \rightsquigarrow_o \top$ *we have* $\mathsf{L}(YZ) \rightsquigarrow_o \top$, *then* $\mathsf{L}(\mathsf{G}XY) \rightsquigarrow_o \top$.

Proof. Follows from definitions and Lemma 7.2.13. □

Lemma 7.2.23. *If* $\mathsf{L}X \rightsquigarrow_o \top$ *and* $\mathsf{F}X\mathsf{L}Y \rightsquigarrow_o \top$ *then* $\mathsf{L}(\Sigma XY) \rightsquigarrow_o \top$.

Proof. Assume $\mathsf{L}X \rightsquigarrow_o \top$ and $\mathsf{F}X\mathsf{L}Y \rightsquigarrow_o \top$. Then $X \rightsquigarrow_{\mathscr{T}} \tau$ for some $\tau \in \mathscr{T}$. Hence $\mathsf{L}(Yd) \rightsquigarrow_o \top$ for $d \in \mathcal{D}_\tau$, by $(\Xi_\top)$, coherence and the definition of F. By $(\mathsf{L}_\top)$, $(\Sigma_{\mathscr{T}})$ and coherence this implies that $\Sigma XY \rightsquigarrow_{\mathscr{T}} \Sigma(\tau, F)$ for appropriate F. Therefore $\mathsf{L}(\Sigma XY) \rightsquigarrow_o \top$ by $(\mathsf{L}_\top)$. □

Lemma 7.2.24. *If* $\mathsf{L}X \rightsquigarrow_o \top$ *and* $\mathsf{F}X\mathsf{L}Y \rightsquigarrow_o \top$ *then* $\mathsf{L}(\mathsf{W}XY) \rightsquigarrow_o \top$.

Proof. Analogous to Lemma 7.2.23. □

Lemma 7.2.25. *If* $\mathsf{L}A \rightsquigarrow_o \top$ *and* $\mathsf{F}A\mathsf{H}X \rightsquigarrow_o \top$ *then* $\mathsf{L}(\Upsilon AX) \rightsquigarrow_o \top$.

Proof. Assume $\mathsf{L}A \rightsquigarrow_o \top$ and $\mathsf{F}A\mathsf{H}X \rightsquigarrow_o \top$. Then $A \rightsquigarrow_{\mathscr{T}} \tau$ for some $\tau \in \mathscr{T}$. Because $\mathsf{F}A\mathsf{H}X \rightsquigarrow_o \top$, for every $d \in \mathcal{D}_\tau$ we have $\mathsf{H}(Xd) \rightsquigarrow_o \top$, i.e., $Xd \rightsquigarrow_o a_d \in \mathcal{D}_o$. By $(\mathcal{F}_{\tau\to o})$ there is $p \in \mathcal{D}_{\tau\to o}$ such that $X \rightsquigarrow_{\tau\to o} p$. Using $(\Upsilon_{\mathscr{T}})$ and coherence we conclude that $\Upsilon AX \rightsquigarrow_{\mathscr{T}} \Upsilon(\tau, S_p)$. Hence $\mathsf{L}(\Upsilon AX) \rightsquigarrow_o \top$ by $(\mathsf{L}_\top)$. □

Lemma 7.2.26. *If* $\mathsf{L}(\mathsf{W}AB) \rightsquigarrow_o \top$ *and for all* X, Y *such that*

- $AX \rightsquigarrow_o \top$ *and*
- $\mathsf{F}(BX)(\mathsf{W}AB)Y \rightsquigarrow_o \top$ *and*
- $\Xi(BX)(\lambda x.Z(Yx)) \rightsquigarrow_o \top$

we have $Z(\sup(\mathsf{W}AB)XY) \rightsquigarrow_o \top$, *then* $\Xi(\mathsf{W}AB)Z \rightsquigarrow_o \top$.

Proof. Assume the antecedent of the above implication. Since $\mathsf{L}(\mathsf{W}AB) \rightsquigarrow_o \top$, there is $\tau = \mathsf{W}(\tau', F) \in \mathscr{T}$ such that $\mathsf{W}AB \rightsquigarrow_{\mathscr{T}} \tau$. By induction on α we show that if $e \in \mathcal{D}_\tau^\alpha$ then $Ze \rightsquigarrow_o \top$. By $(\Xi_\top)$ this will imply that $\Xi(\mathsf{W}AB)Z \rightsquigarrow_o \top$. So let $e = \langle d, f\rangle \in \mathcal{D}_\tau^\alpha$. Because $\mathsf{W}AB \rightsquigarrow_{\mathscr{T}} \tau = \mathsf{W}(\tau', F)$ and $d \in \mathcal{D}_{\tau'}$, we have $A \rightsquigarrow_{\mathscr{T}} \tau'$ and $Bd \rightsquigarrow_{\mathscr{T}} F(d)$. By Lemma 7.2.13 we also have $Ad \rightsquigarrow_o \top$. Let $a \in \mathcal{D}_{F(d)}$. Then $f^{\mathcal{F}}(a) \in \mathcal{D}_\tau^{<\alpha}$, so $Z(f^{\mathcal{F}}(a)) \rightsquigarrow_o \top$ by the inductive hypothesis. Since $a \in \mathcal{D}_{F(d)}$ was arbitrary, this implies $\Xi(Bd)(\lambda x.Z(fx)) \rightsquigarrow_o \top$. Since for every $a \in \mathcal{D}_{F(d)}$ we have $\mathsf{W}AB(fa) \rightsquigarrow_o \top$ by Lemma 7.2.13, and $Bd \rightsquigarrow_{\mathscr{T}} F(d)$, we also have $\mathsf{F}(Bd)(\mathsf{W}AB)f \rightsquigarrow_o \top$. Hence $Z(\sup(\mathsf{W}AB)df) \rightsquigarrow_o \top$. Since $\sup(\mathsf{W}AB)df \to \langle d, f\rangle$ by (sup), we conclude $Ze \equiv Z\langle d, f\rangle \rightsquigarrow_o \top$ by coherence. □

Lemma 7.2.27. *If* $\mathsf{L}A \rightsquigarrow_o \top$ *and for all* Z *with* $AZ \rightsquigarrow_o \top$ *we have* $\mathsf{Q}_L(BZ)(XZ)(YZ) \rightsquigarrow_o \top$*, then* $\mathsf{Q}_L(\mathsf{G}AB)XY \rightsquigarrow_o \top$*.*

Proof. The proof is similar to the proof of Lemma 6.2.18. Recall that

$$\mathsf{Q}_L AXY =_\beta \Xi(\mathsf{F}A\mathsf{H})(\lambda p.\neg(pX) \vee pY).$$

Suppose

(1) $\mathsf{L}A \rightsquigarrow_o \top$ and

(2) for every Z with $AZ \rightsquigarrow_o \top$ we have $\mathsf{Q}_L(BZ)(XZ)(YZ) \rightsquigarrow_o \top$.

Since $\mathsf{L}A \rightsquigarrow_o \top$, we have $A \rightsquigarrow_{\mathscr{T}} \tau_1$ for some $\tau_1 \in \mathscr{T}$ by $(\mathsf{L}_\top)$ in Definition 7.2.3. We need to show

$$(\star) \quad \Xi(\mathsf{F}(\mathsf{G}AB)\mathsf{H})(\lambda p.\neg(pX) \vee pY).$$

First assume $\mathcal{D}_{\tau_1} = \emptyset$. Then $\mathsf{G}AB \rightsquigarrow_{\mathscr{T}} \tau = \mathsf{G}(\tau_1, F)$ by $(\mathsf{G}_{\mathscr{T}})$, where F is the empty function. Then $\mathsf{F}(\mathsf{G}AB)\mathsf{H} \rightsquigarrow_{\mathscr{T}} \tau \to o$. Let $f \in \mathcal{D}_\tau$ be the only element of $\mathcal{D}_\tau$ – the empty function. Note that because $\mathcal{D}_{\tau_1} = \emptyset$, by $(\mathcal{F}_\tau)$ we have $Z \succ_\tau f$ for an arbitrary term Z. Let $p \in \mathcal{D}_{\tau\to o}$. It suffices to show that $pX \rightsquigarrow_o \bot$ or $pY \rightsquigarrow_o \top$, and then $(\star)$ follows by definitions. We have $pf \rightsquigarrow_o \top$ or $pf \rightsquigarrow_o \bot$. Since $X \succ_\tau f$ and $Y \succ_\tau f$ we obtain $pX \rightsquigarrow_o \bot$ or $pY \rightsquigarrow_o \top$ by invariance.

Now assume $\mathcal{D}_{\tau_1} \neq \emptyset$. Then there is $d \in \mathcal{D}_{\tau_1}$, and by Lemma 7.2.13 we have $Ad \rightsquigarrow_o \top$. Thus $\mathsf{Q}_L(Bd)(Xd)(Yd) \rightsquigarrow_o \top$ by (2), so there is $\tau_d \in \mathscr{T}$ with $Bd \rightsquigarrow_{\mathscr{T}} \tau_d$ by $(\mathsf{L}_\top)$, $(\Xi_\top)$, $(\mathsf{F}_{\mathscr{T}})$ and coherence. Since $A \rightsquigarrow_{\mathscr{T}} \tau_1$ and for every $d \in \mathcal{D}_{\tau_1}$ we have $Bd \rightsquigarrow_{\mathscr{T}} \tau_d$, by $(\mathsf{G}_{\mathscr{T}})$ we conclude $\mathsf{G}AB \rightsquigarrow_{\mathscr{T}} \tau$ where $\tau = \mathsf{G}(\tau_1, F_B) \in \mathscr{T}$ and $F_B(d) = \tau_d$ for $d \in \mathcal{D}_{\tau_1}$.

We show that there is $f \in \mathcal{D}_\tau$ with $X \rightsquigarrow_\tau f$ and $Y \rightsquigarrow_\tau f$. Let $d \in \mathcal{D}_{\tau_1}$. Then $Ad \rightsquigarrow_o \top$ by Lemma 7.2.13, because $A \rightsquigarrow_{\mathscr{T}} \tau_1$. So $\mathsf{Q}_L(Bd)(Xd)(Yd) \rightsquigarrow_o \top$ and by Lemma 7.2.16 there is $b_d \in \mathcal{D}_{\tau_d}$ with $Xd \rightsquigarrow_{\tau_d} b_d$ and $Yd \rightsquigarrow_{\tau_d} b_d$. Thus by $(\mathcal{F}_\tau)$ we may take $f \in \mathcal{D}_\tau$ with $f^{\mathcal{F}}(d) \equiv b_d$ for $d \in \mathcal{D}_{\tau_1}$.

Since $\mathsf{G}AB \rightsquigarrow_{\mathscr{T}} \tau$, we have $\mathsf{F}(\mathsf{G}AB)\mathsf{H} \rightsquigarrow_{\mathscr{T}} \tau \to o$. Let $p \in \mathcal{D}_{\tau\to o}$. We have $pf \rightsquigarrow_o \top$ or $pf \rightsquigarrow_o \bot$. Therefore $\neg(pf) \vee pf \rightsquigarrow_o \top$. By invariance $\neg(pX) \vee pY \rightsquigarrow_o \top$. Since $p \in \mathcal{D}_\tau$ was arbitrary, we obtain $(\star)$ by $(\Xi_\top)$ and coherence. □

Lemma 7.2.28. *If* $X, Y \rightsquigarrow_o \top$ *or* $X, Y \rightsquigarrow_o \bot$ *then* $\mathsf{Q}_L\mathsf{H}XY \rightsquigarrow_o \top$*.*

Proof. The proof is analogous to the proof of Lemma 6.2.19. □

Lemma 7.2.29. *If the following conditions hold*

(1) $\mathsf{Q}_L A(\pi_1 X)(\pi_1 Y) \rightsquigarrow_o \top$,

(2) $\mathsf{Q}_L(B(\pi_1 X))(\pi_2 X)(\pi_2 Y) \rightsquigarrow_o \top$,

(3) $\mathsf{L}(\Sigma AB) \rightsquigarrow_o \top$,

then $\mathsf{Q}_L(\Sigma AB)XY \rightsquigarrow_o \top$*.*

Proof. Assume (1)−(3). Since $\mathsf{Q}_L A(\pi_1 X)(\pi_1 Y) \rightsquigarrow_o \top$, there is $\tau \in \mathscr{T}$ with $A \rightsquigarrow_{\mathscr{T}} \tau$, by $(\Xi_\top)$, $(\mathsf{L}_\top)$, $(\mathsf{F}_{\mathscr{T}})$ and coherence. By Lemma 7.2.16 there is $d \in \mathcal{D}_\tau$ with $\pi_1 X \rightsquigarrow_\tau d$ and $\pi_1 Y \rightsquigarrow_\tau d$. Since $\mathsf{L}(\Sigma AB) \rightsquigarrow_o \top$, there is $\tau = \Sigma(\tau_1, F) \in \mathscr{T}$ with $\Sigma AB \rightsquigarrow_{\mathscr{T}} \tau$. Then $Bd \rightsquigarrow_{\mathscr{T}} \tau_2 = F(d)$, so also $B(\pi_1 X) \rightsquigarrow_{\mathscr{T}} \tau_2$ by invariance. By (2) and Lemma 7.2.16 there is $b \in \mathcal{D}_{\tau_2}$ with $\pi_2 X \rightsquigarrow_{\tau_2} b$ and $\pi_2 Y \rightsquigarrow_{\tau_2} b$. Now by (π_τ^Σ) we conclude $X \rightsquigarrow_\tau \langle d, b\rangle$ and $Y \rightsquigarrow_\tau \langle d, b\rangle$. Since, as is easily checked, $\mathsf{Q}_L(\Sigma AB)\langle d, b\rangle\langle d, b\rangle \rightsquigarrow_o \top$, by invariance $\mathsf{Q}_L(\Sigma AB)XY \rightsquigarrow_o \top$. □

Lemma 7.2.30. *If* $\mathsf{Q}_L AXX' \rightsquigarrow_o \top$ *and* $\mathsf{Q}_L(\mathsf{F}(BX)(\mathsf{W}AB))YY' \rightsquigarrow_o \top$ *and* $\mathsf{L}(\mathsf{W}AB) \rightsquigarrow_o \top$, *then* $\sup(\mathsf{W}AB)XY = \sup(\mathsf{W}AB)X'Y'$.

Proof. Assume the antecedent of the implication. Since $\mathsf{L}(\mathsf{W}AB) \rightsquigarrow_o \top$, we have $\mathsf{W}AB \rightsquigarrow_{\mathscr{T}} \mathsf{W}(\tau, F)$. Then $A \rightsquigarrow_{\mathscr{T}} \tau$. Because $\mathsf{Q}_L AXX' \rightsquigarrow_o \top$, by Lemma 7.2.16 there is $d \in \mathcal{D}_\tau$ such that $X \rightsquigarrow_\tau d$ and $X' \rightsquigarrow_\tau d$. Then also $Bd \rightsquigarrow_{\mathscr{T}} F(d)$, so $BX \rightsquigarrow_{\mathscr{T}} F(d)$ by invariance. Thus $\mathsf{F}(BX)(\mathsf{W}AB) \rightsquigarrow_{\mathscr{T}} F(d) \to \mathsf{W}(\tau, F)$. Hence, because $\mathsf{Q}_L(\mathsf{F}(BX)(\mathsf{W}AB))YY' \rightsquigarrow_o \top$, by Lemma 7.2.16 there is $f \in \mathcal{D}_{F(d)\to\mathsf{W}(\tau,F)}$ such that $Y \rightsquigarrow_{F(d)\to\mathsf{W}(\tau,F)} f$ and $Y' \rightsquigarrow_{F(d)\to\mathsf{W}(\tau,F)} f$. Hence by (sup) we conclude $\sup(\mathsf{W}AB)XY \to \langle d, f\rangle$ and $\sup(\mathsf{W}AB)X'Y' \to \langle d, f\rangle$. □

Definition 7.2.31. Define $\mathcal{M} = \langle \mathcal{C}, I, \mathcal{T}, \mathcal{F}\rangle$ where:

- $\mathcal{C}$ is the combinatory algebra constructed from the R-equality equivalence classes of terms, with $\mathsf{k} = [\mathsf{K}]$, $\mathsf{s} = [\mathsf{S}]$, $\Xi = [\Xi]$, etc., where by $[X]$ we denote the equivalence class of X,
- I is defined by $I(c) = [c]$ for $c \in \Sigma$,
- $\mathcal{T} = \{[X] \mid X \rightsquigarrow_o \top\}$,
- $\mathcal{F} = \{[X] \mid X \rightsquigarrow_o \bot\}$.

Theorem 7.2.32. *The structure* $\mathcal{M}$ *from Definition 7.2.31 is an* $\mathcal{I}^+$*-model.*

Proof. We need to check that $\mathcal{M}$ satisfies the conditions 1-29 from Definition 7.1.8. The conditions 1-14 follow from definitions, coherence and Lemma 7.2.14. Conditions 15-17 follow from Lemma 7.2.19, conditions 18-19 from Lemma 7.2.20, condition 20 from Lemma 7.2.21, condition 21 from Lemma 7.2.22, condition 22 from Lemma 7.2.23, condition 23 from Lemma 7.2.24, condition 24 from Lemma 7.2.25, condition 25 from Lemma 7.2.26, condition 26 from Lemma 7.2.27, condition 27 from Lemma 7.2.28, condition 28 follows from Lemma 7.2.29, and condition 29 from Lemma 7.2.30. □

Corollary 7.2.33. *The system* $\mathcal{I}^+$ *is consistent, i.e.,* $\nvdash_{\mathcal{I}^+} \bot$.

Conclusion

We introduced four classical systems of illative combinatory logic: the propositional system $\mathcal{I}$Kp, the first-order system $\mathcal{I}$K, the higher-order system $e\mathcal{I}\mathrm{K}\omega$, and the extended higher-order system $\mathcal{I}^+$. We also investigated the intuitionistic variant $\mathcal{I}$Jp (resp. $\mathcal{I}$J) of $\mathcal{I}$Kp (resp. $\mathcal{I}$K), and an intensional variant $\mathcal{I}\mathrm{K}\omega$ of $e\mathcal{I}\mathrm{K}\omega$. For each system a semantics was presented and the systems were shown sound w.r.t. the corresponding semantics. The systems $\mathcal{I}$Jp, $\mathcal{I}$Kp and $\mathcal{I}$J were also shown to be complete. The system $\mathcal{I}$K was shown complete w.r.t. a sligthly less natural class of models. We proved all systems consistent by model constructions.

We also investigated some translations of traditional systems of logic into corresponding illative systems. We proved all those translations to be sound, i.e., if a judgement of a traditional system is provable, then so is its translation. For $\mathcal{I}$Jp, $\mathcal{I}$Kp, $\mathcal{I}$J and $\mathcal{I}$K we also showed the translations complete, i.e., if the translation of a judgement is provable, then so is the original judgement. For $\mathcal{I}\mathrm{K}\omega$ and $e\mathcal{I}\mathrm{K}\omega$ we derived a limited completeness result: if a translated judgement of higher-order logic is provable in $e\mathcal{I}\mathrm{K}\omega$ then it is valid in all standard models for higher-order logic. The proofs of most of these results were done semantically, by showing a truth-preserving transformation of models of illative systems into models of corresponding traditional systems, and vice versa.

Some of the systems shown consistent in the present work are much stronger than the systems of [BBD93, DBB98a, DBB98b]. In particular, the system $e\mathcal{I}\mathrm{K}\omega$ essentially incorporates full extensional classical higher-order logic. The strongest of our systems $\mathcal{I}^+$ extends $e\mathcal{I}\mathrm{K}\omega$ with dependent function types, dependent sums, subtypes and W-types.

The system $\mathcal{I}^+$ is rich enough to interpret a great deal of mathematics. Many common type-theoretic constructions are possible. Using dependent sums one may define finite products and (non-dependent) disjoint sums. Using W-types one may define inductive types, including the type of natural numbers. The derived induction principles associated with inductive types are unrestricted, i.e., it is possible to apply inductive reasoning to terms whose types have not yet been established, thus for instance enabling reasoning about types of terms by induction.

From a foundational viewpoint, what distinguishes illative combinatory logic is that it is extremely simple and it assumes as primitive the notion of self-applicable function-in-intension (operation). The simplicity of illative combinatory logic is a consequence of the fact that it was invented with the intention of analysing prelogic. According to Curry, the aim of illative combinatory logic is not merely to provide an alternative foundational system for mathematics, which would compete with the theory of types, set theory, etc. In Curry's view, combinatory logic concerns itself with the ultimate foundations. Its purpose is the analysis of certain notions of such a basic character that they are taken for granted in most other systems of logic. These are, above all, the analysis of the process of substitution, and also the classification of objects into types or categories. Such notions constitute a prelogic. Although very basic and generally presupposed, these notions are not simple and thus they merit further investigation. Moreover, an analysis of prelogic may shed some light on the sources of paradoxes.

From the point of view of computer science, an interesting feature of illative systems is that by basing on the untyped lambda-calculus (combinatory logic) they incorporate general recursion into the logic. Using illative-like systems may thus be a viable approach to the problem of handling unrestricted recursion in interactive theorem provers. An advantage of illative systems is that no justifications are needed for formulating unrestricted recursive definitions. One may just introduce a possibly non-well-founded recursive function definition and start reasoning about it within the logic.

To avoid inconsistency some inference rules need to be restricted by adding premises which essentially state that some terms are "propositions". To be able to derive that some terms are propositions, illative systems include certain "typing rules", i.e., rules for reasoning about which types (categories) a term belongs to. In contrast to traditional systems, however, these rules are internal to the system. The functions do not need to be "typed" a priori, but reasoning about "types" may be interleaved with other reasoning. For instance, one may show typability by induction.

The "typing rules" in illative systems are of such a character that in most cases deriving the additional premises is straightforward. In particular, the soundness of translations of traditional systems of logic into illative combinatory logic shows that additional premises in introduction rules hold as long as we deal only with terms which are translations of terms or formulas of a traditional system. Explicitly deriving the additional premises may be needed only when dealing with terms which do not have direct counterparts in traditional systems.

Furthermore, the "typing rules" are similar to rules in traditional type systems. In fact, these rules are usually generalisations of traditional typing rules. Therefore, in a machine implementation of illative logic, it may be possible to adapt standard type checking or type inference algorithms to obtain algorithms which, in common cases, automatically produce a derivation establishing which type a given term belongs to, and thus dispose of the additional premises in introduction rules.

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