Lecture 2: The Curry-Howard isomorphism

Łukasz Czajka

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NOTE: Propositional variables may be represented by nullary (0-ary) predicates.

Predicate logic: syntax

$$P \to Q \to R$$

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$$P \to (Q \to R)$$

Predicate logic: syntax

$$P \vee Q \rightarrow R$$

Predicate logic: syntax

$$(P \vee Q) \to R$$

Predicate logic: syntax

$$\forall x R(x) \vee \forall x \neg S(x)$$

Predicate logic: syntax

$$(\forall x R(x)) \vee (\forall x \neg S(x))$$

Predicate logic: the "dot" notation

$$\forall x.R(x) \vee S(x)$$

Predicate logic: the “dot” notation

$$\forall x(R(x) \vee S(x))$$

Predicate logic: free variables

$$Q(x) \vee \forall x \exists y R(x, y, z)$$

Predicate logic: free variables

$$Q(\mathbf{x}) \vee \forall x \exists y R(x, y, \mathbf{z})$$

Predicate logic: variable scopes

$$\forall x.\exists x R(x, x) \vee Q(x)$$

Predicate logic: variable scopes

$$\forall x.(\exists y R(y, y)) \lor Q(x)$$

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NOTE: For propositional formulas (without quantifiers and with only nullary predicates) the above semantics is the same as propositional truth-table semantics.

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- But in general we don't know which of $\varphi, \neg\varphi$ holds!
- Similarly, we may be able to classicaly prove $\exists x \varphi$ but still not be able to provide any concrete element for which φ holds.

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Theorem

There exist irrational numbers a, b such that a^b is rational.

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Proof.

The number $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational. If $\sqrt{2}^{\sqrt{2}}$ is rational then take $a = b = \sqrt{2}$. If $\sqrt{2}^{\sqrt{2}}$ is irrational then $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$ is rational, so take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. $\square$

Intuitionistic logic

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- Intuitionistic (constructive) logic is about constructibility.

The three constructivists: Brouwer, Heyting, Kolmogorov

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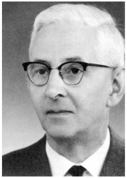

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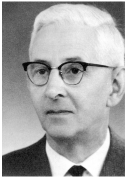

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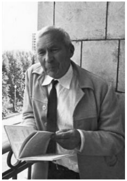

"In addition to theoretical logic, which systematises proof schemata for theoretical truths, one can systematise proof schemata for solutions to problems. (...) Intuitionistic logic should rather be called the calculus of problems, since its objects are in reality problems, rather than theoretical propositions." A. Kolmogorov

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$\neg\varphi$ is an abbreviation for $\varphi \to \perp$.

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The BHK interpretation

Excluded middle:

$$P \vee \neg P$$

In general, we cannot provide for an arbitrary P either a proof of P or a function which transforms a proof of P into a proof of $\perp$. It seems then that $P \vee \neg P$ is not intuitionistically provable.

The BHK interpretation

Peirce's law:

$$((P \to Q) \to P) \to P$$

If we are given just a function which transforms a proof of $P \to Q$ into a proof of P, then there doesn't seem to be any way of using this function to obtain a proof of P. To use the function, we would need to construct a proof of $P \to Q$, i.e., a function which converts any proof of P into a proof of Q. This does not seem possible in general. Hence, it seems Peirce's law is not intuitionistically provable.

The BHK interpretation

NOTE: The BHK interpretation is informal, so the above arguments do not conclusively establish if the formulas are intuitionistically provable. Such arguments may nonetheless be helpful to quickly determine whether intuitionistic provability is plausible.

Curry-Howard isomorphism

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- Formulas are types.
- Proofs are lambda-terms (roughly: total functional programs).

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- $\perp$ is an atomic formula and we define $\neg\varphi \equiv \varphi \to \perp$.
- Aside of these changes, the formulas (types) are exactly the formulas of first-order logic.

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- We now present in detail the proof terms and the typing rules.

Intermission: derivation rules

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- If we have derived the judgements $J_1, \ldots, J_n$ and the side condition S holds, then we can derive the judgement J.

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- Sometimes we write the side condition(s) above the line together with the judgements $J_1, \dots, J_n$.

Intermission: derivation trees

$$\dfrac{\dfrac{\dfrac{}{J_3} \qquad \dfrac{\dfrac{}{J_5}}{J_4}}{J_1} \qquad \dfrac{}{J_2}}{J}$$

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- At the leaves of the tree we need rules with no judgements above the line.

Curry-Howard isomorphism

A proof of $\varphi_1 \to \varphi_2$ is a method (constructive function) which transforms any proof of φ_1 into a proof of φ_2.

$$\frac{\Gamma, X : \varphi_1 \vdash M : \varphi_2}{\Gamma \vdash (\lambda X : \varphi_1 . M) : \varphi_1 \to \varphi_2}\ (\to\mathrm{I}) \qquad \frac{\Gamma \vdash M_1 : \varphi \to \psi \quad \Gamma \vdash M_2 : \varphi}{\Gamma \vdash M_1 M_2 : \psi}\ (\to\mathrm{E})$$

introduction (how to prove) elimination (how to use)

Curry-Howard isomorphism

A proof of $\varphi_1 \wedge \varphi_2$ consists of a proof of φ_1 and a proof of φ_2.

$$\frac{\Gamma \vdash M_1 : \varphi_1 \quad \Gamma \vdash M_2 : \varphi_2}{\Gamma \vdash (M_1, M_2) : \varphi_1 \wedge \varphi_2}\ (\wedge \mathrm{I})$$

$$\frac{\Gamma \vdash M : \varphi_1 \wedge \varphi_2 \quad \Gamma, X_1 : \varphi_1, X_2 : \varphi_2 \vdash N : \psi}{\Gamma \vdash (\texttt{case}\, M\, \texttt{of}\, (X_1, X_2) \Rightarrow N) : \psi}\ (\wedge \mathrm{E})$$

Curry-Howard isomorphism

A proof of $\varphi_1 \vee \varphi_2$ consists of an indicator $i \in \{1, 2\}$ and a proof of φ_i.

$$\frac{\Gamma \vdash M : \varphi_1}{\Gamma \vdash \texttt{inl}\, M : \varphi_1 \vee \varphi_2}\ (\vee\mathrm{I}_1) \qquad \frac{\Gamma \vdash M : \varphi_2}{\Gamma \vdash \texttt{inr}\, M : \varphi_1 \vee \varphi_2}\ (\vee\mathrm{I}_2)$$

$$\frac{\Gamma \vdash M : \varphi_1 \vee \varphi_2 \quad \Gamma, X_1 : \varphi_1 \vdash N_1 : \psi \quad \Gamma, X_2 : \varphi_2 \vdash N_2 : \psi}{\Gamma \vdash (\texttt{case}\, M\, \texttt{of}\, \texttt{inl}\, X_1 \Rightarrow N_1 \mid \texttt{inr}\, X_2 \Rightarrow N_2) : \psi}\ (\vee\mathrm{E})$$

Curry-Howard isomorphism

There is no proof of $\perp$.

$$\frac{\Gamma \vdash M : \perp}{\Gamma \vdash (\mathtt{case}_\psi\, M) : \psi}\ (\perp\mathrm{E})$$

Curry-Howard isomorphism

A proof of $\forall x : A.\varphi$ is a method (constructive function) which transforms any object t in A into a proof of $\varphi(t)$.

$$\frac{\Gamma \vdash M : \varphi \quad x : A \quad x \notin \mathrm{FV}(\Gamma)}{\Gamma \vdash (\lambda x : A.M) : \forall x : A.\varphi}\ (\forall\mathrm{I}) \qquad \frac{\Gamma \vdash M : \forall x : A.\varphi \quad t : A}{\Gamma \vdash Mt : \varphi[t/x]}\ (\forall\mathrm{E})$$

Note: $\mathrm{FV}(\Gamma)$ is the set of all object variables occuring free in one of the formulas declared in Γ

Curry-Howard isomorphism

A proof of $\exists x : A.\varphi$ consists of an object t in A and a proof of $\varphi(t)$.

$$\frac{\Gamma \vdash M : \varphi[t/x] \quad t : A \quad x : A}{\Gamma \vdash [t, M] : \exists x : A.\varphi}\ (\exists\mathrm{I})$$

$$\frac{\Gamma \vdash M : \exists x : A.\varphi \quad \Gamma, X : \varphi \vdash N : \psi \quad x \notin \mathrm{FV}(\Gamma, \psi)}{\Gamma \vdash (\mathtt{case}\, M\, \mathtt{of}\, [x, X] \Rightarrow N) : \psi}\ (\exists\mathrm{E})$$

Note: $\mathrm{FV}(\Gamma, \psi)$ is the set of all object variables occuring free in one of the formulas declared in Γ or in ψ

Predicate logic: intuitionistic natural deduction

$$\frac{}{\Gamma, X : \varphi \vdash X : \varphi}\ (\mathrm{Ax})$$

$$\frac{\Gamma, X : \varphi_1 \vdash M : \varphi_2}{\Gamma \vdash (\lambda X : \varphi_1 . M) : \varphi_1 \to \varphi_2}\ (\to\mathrm{I}) \qquad \frac{\Gamma \vdash M_1 : \varphi \to \psi \quad \Gamma \vdash M_2 : \varphi}{\Gamma \vdash M_1 M_2 : \psi}\ (\to\mathrm{E})$$

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$$\frac{\Gamma \vdash M : \bot}{\Gamma \vdash (\mathsf{case}_\psi\ M) : \psi}\ (\bot\mathrm{E})$$

$$\frac{\Gamma \vdash M : \varphi \quad x : A \quad x \notin \mathrm{FV}(\Gamma)}{\Gamma \vdash (\lambda x : A . M) : \forall x : A . \varphi}\ (\forall\mathrm{I}) \qquad \frac{\Gamma \vdash M : \forall x : A . \varphi \quad t : A}{\Gamma \vdash M t : \varphi[t/x]}\ (\forall\mathrm{E})$$

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However, with such a “naive” extension we lose the correspondence between proofs and functional programs.

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Theorem

$\Gamma \vdash_c M : \varphi$ *for some* M *if and only if* $|\Gamma| \models_c \varphi$.

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- A formula φ is a (classical) <u>semantic consequence</u> of a set of formulas Δ, denoted $\Delta \models_c \varphi$, if for every interpretation I such that $I \models \psi$ for each $\psi \in \Delta$, we have $I \models \varphi$.
- Let $|\Gamma|$ denote the set of formulas declared in the context Γ.
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Conclusion: intuitionistic logic is a subsystem of classical logic with a "natural" computational interpretation.