Lecture 4: Dependent types and the Calculus of Constructions

Łukasz Czajka

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- Full dependent types abolish the a priori distinction between proof terms (proofs) and object terms (programs).
- It becomes possible to quantify over proofs (which are programs), and proofs (programs) may occur in types (formulas).

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- The type of the result depends on the value of the argument!
- $\sigma \to \tau$ is a special case of $\forall x : \sigma.\tau$ when $x \notin \mathrm{FV}(\tau)$ (i.e. x does not occur free in τ).

Intermission: the simply-typed lambda-calculus

Simple types: $\mathcal{T} ::= \mathcal{B} \mid \mathcal{T} \to \mathcal{T}$ where $\mathcal{B}$ is a fixed finite set of type constants.

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- Uniqueness of normal forms: if t_1, t_2 are in β-normal form and $t_1 =_\beta t_2$, then $t_1 = t_2$.
- Exercise: β-equality on simply-typed terms is decidable.

Intermission: the simply-typed lambda-calculus

Let's assume the elements of $\mathcal{B}$ are ordinary variables and $t_1 \to t_2$ is just another form of terms. Let $*$ be the universe of types.

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Let the contexts be sequences instead of sets.

$$\frac{\Gamma \vdash \tau : * \quad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : \tau \vdash x : \tau} \qquad \frac{\Gamma \vdash t : \tau \quad \Gamma \vdash \sigma : * \quad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : \sigma \vdash t : \tau}$$

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Dependent types: the system λP

- A term t, τ, σ is a variable x, y, z, α, β, a universe $\mathcal{u} \in \mathcal{U}$, an application $t_1 t_2$, a lambda-abstraction $\lambda x : \tau . t$, or a dependent function type $\forall x : \sigma . \tau$.

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- A judgement has the form $\Gamma \vdash t : \tau$ with Γ context, t, τ terms.

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- β-reduction: $(\lambda x : \tau.t)t' \to_\beta t[t'/x]$.
- η-reduction: $\lambda x : \tau.tx \to_\eta t$ if $x \notin \mathrm{FV}(t)$.
- Definitional equality $\equiv$ is defined as $\beta\eta$-equality.

Dependent types: the system λP

$$\frac{(\mathcal{U}_1, \mathcal{U}_2) \in \mathcal{A}}{\langle\rangle \vdash \mathcal{U}_1 : \mathcal{U}_2}$$

$$\frac{\Gamma \vdash \tau : \mathcal{U} \quad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : \tau \vdash x : \tau} \qquad \frac{\Gamma \vdash t : \tau \quad \Gamma \vdash \sigma : \mathcal{U} \quad x \notin \mathrm{dom}(\Gamma)}{\Gamma, x : \sigma \vdash t : \tau}$$

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$$\frac{\Gamma \vdash \tau : \mathcal{U}_1 \quad \Gamma, x : \tau \vdash \sigma : \mathcal{U}_2 \quad (\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3) \in \mathcal{R}}{\Gamma \vdash (\forall x : \tau.\sigma) : \mathcal{U}_3}$$

$$\frac{\Gamma \vdash t : \tau \quad \Gamma \vdash \tau' : \mathcal{U} \quad \tau \equiv \tau'}{\Gamma \vdash t : \tau'}$$

- Universes: $\mathcal{U} = \{*, \Box\}$.
- Axioms: $\mathcal{A} = \{(*, \Box)\}$.
- Rules: $\mathcal{R} = \{(*, *, *), (*, \Box, \Box)\}$.

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- Unless stated otherwise, we consider only legal terms and contexts (i.e. those which appear in some derivable judgement).

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  - E.g.: $\alpha : *, P : \alpha \to \alpha \to * \vdash (\lambda x : \alpha . P x x) : \alpha \to * : \Box$.
- The universe of types is a kind: $* : \Box$ because $(*, \Box) \in \mathcal{A}$. So each type (formula/proposition) is a type constructor (nullary predicate).

Dependent types: λP

$\Box$					
$*$			$\alpha \to *$	$(\alpha \to \alpha) \to \alpha \to *$	$\ldots$
α	$(\forall x : \alpha . Px) \to Py$	$\ldots$	$\lambda x : \alpha . Px$	$\lambda f : \alpha \to \alpha . \lambda x : \alpha . P(fx)$	$\ldots$
y	$\lambda f : \forall x : \alpha . Px . fy$	$\ldots$	—		

In the context: $\alpha : *, P : \alpha \to *, y : \alpha, p : \forall x : \alpha . Px$.

Dependent types: rules of λP

Objects depend on objects: $(*, *, *) \in \mathcal{R}$.

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- $\Gamma, x : \alpha \vdash x : \alpha$.
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- For this we need $\alpha : * \vdash (\alpha \to *) : \Box$.

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Dependent types: rules of λP

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The rule $(*, \Box, \Box)$ allows us to have "predicates" in the context (but not to quantify over them). λP is essentially a "first-order" system.

λP vs first-order logic

Consider the universal-implicational fragment FOL$\forall\to$ of the system of inituitionistic first-order logic from the second lecture.

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Assuming the proof and object variables and the domains of FOL$\forall\to$ are variables in λP, we define a translation from FOL$\forall\to$ to λP:

- $\lceil X \rceil = X$, $\lceil x \rceil = x$, $\lceil M_1 M_2 \rceil = \lceil M_1 \rceil \lceil M_2 \rceil$, $\lceil Mt \rceil = \lceil M \rceil \lceil t \rceil$, $\lceil \lambda x : A . M \rceil = \lambda x : A . \lceil M \rceil$, $\lceil \lambda X : \varphi . M \rceil = \lambda X : \lceil \varphi \rceil . \lceil M \rceil$.
- $\lceil A \rceil = A$, $\lceil \varphi \to \psi \rceil = \lceil \varphi \rceil \to \lceil \psi \rceil$, $\lceil \forall x : A . \varphi \rceil = \forall x : A . \lceil \varphi \rceil$.

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$$\lceil \Gamma \vdash M : \varphi \rceil =$$
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Theorem (Soundness of translation from FOL$\forall\rightarrow$ to λP)

If $\Gamma \vdash M : \varphi$ *is derivable in FOL*$\forall\rightarrow$ *then* $\lceil \Gamma \vdash M : \varphi \rceil$ *is derivable in* λP.

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- Domains of quantifications may be empty, in contrast to "ordinary" first-order logic where they are implicitly assumed to be non-empty. E.g.: $(\forall x : \tau.\psi) \to \psi$ with $x \notin \mathrm{FV}(\psi)$ is not inhabited unless we can construct an element of τ, even though the corresponding first-order formula $\forall x\psi \to \psi$ is an intuitionistic tautology when $x \notin \mathrm{FV}(\psi)$.

Pure Type Systems

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- The (many-step) $\beta\eta$-reduction relation $\to^*_{\beta\eta}$ is the transitive-reflexive closure of $\to_{\beta\eta} = \to_\beta \cup \to_\eta$.

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- A PTS is strongly (resp. weakly) normalising if every legal term is strongly (resp. weakly) normalising.

Exercise: postponement of η-reduction

Proposition

If t is strongly (resp. weakly) β-normalising, then it is strongly (resp. weakly) $\beta\eta$-normalising.

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Proof (sketch).

For strong normalisation, show that if $t_1 \to_\eta t_2 \to_\beta t_3$ then there is t' with $t_1 \to_\beta^+ t' \to_\eta^* t_3$. For weak normalisation, it suffices to prove that η-reduction is normalising and that η-reducing a β-normal form produces a β-normal form. □

Pure Type Systems: properties

Theorem (Subject reduction for β)

In any PTS, if $\Gamma \vdash t : \tau$ *and* $t \to_\beta^* t'$ *then* $\Gamma \vdash t' : \tau$.

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Theorem (Uniqueness of normal forms)

In any PTS, if t_1, t_2 *are* legal *(well-typed)* $\beta\eta$*-normal forms such that* $t_1 =_{\beta\eta} t_2$, *then* $t_1 = t_2$.

Pure Type Systems: benefits of normalisation

Theorem (Decidability of type checking)

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Theorem (Consistency)

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- Let $\texttt{Type}_0 = \texttt{Set}$.

Universes: rules

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- $(\mathcal{U}, \mathtt{Prop}, \mathtt{Prop}) \in \mathcal{R}$ for any universe $\mathcal{U}$.

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Definition

A universe $\mathcal{U}_1$ with $\mathcal{U}_1 : \mathcal{U}_2$ is impredicative if $(\mathcal{U}_2, \mathcal{U}_1, \mathcal{U}_1) \in \mathcal{R}$, i.e.,

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- For a predicative universe $\mathcal{U}_1$ with $\mathcal{U}_1 : \mathcal{U}_2$ we have $(\mathcal{U}_2, \mathcal{U}_1, \mathcal{U}_2) \in \mathcal{R}$ instead, i.e.,

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The system λU^- is inconsistent, i.e., $\vdash t : \forall p : *.p$ is derivable for some t.

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Corollary

Any PTS with $(*, *) \in \mathcal{A}$ and $(*, *, *) \in \mathcal{R}$ is inconsistent.

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- But in Coq impredicative `Set` is inconsistent with the combination of classical logic and the axiom of choice!

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- One consequence of subtyping: subject reduction for η-reduction fails – η-expansion on legal terms is considered instead.

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- *In λC, the excluded middle axiom and the axiom of choice together imply proof irrelevance.*

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- *In λC, the excluded middle axiom and the axiom of choice together imply proof irrelevance.*

Decidability of equality (excluded middle for equality):

$$\forall A : \texttt{Type}. \forall x y : A. x = y \vee x \neq y.$$

Proof irrelevance

Proof irrelevance axiom:

$$\forall A : \texttt{Prop}. \forall p_1 p_2 : A. p_1 = p_2$$

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Theorem

In Coq, proof irrelevance and the axiom of choice together imply decidability of equality.

Axioms

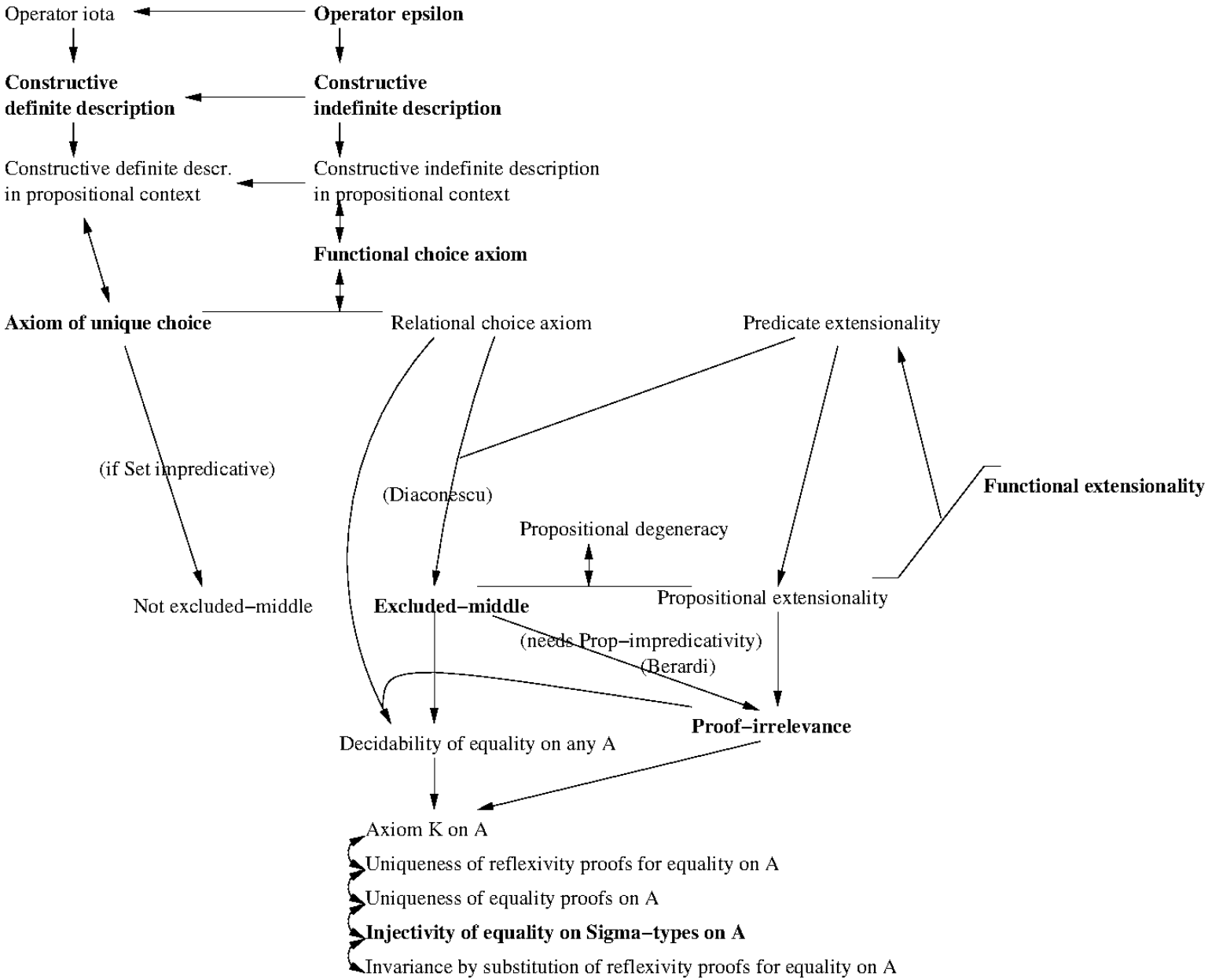

Image source: `https://github.com/coq/coq/wiki/The-Logic-of-Coq`