Lecture 6: Equality

Łukasz Czajka

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- ζ-equality:

  $$(\texttt{let x := s in t}) =_\zeta \texttt{t[s/x]}$$

Conversion rule

Coq's conversion relation $\leq$ includes definitional equality and subtyping between universes.

$$\frac{\Gamma \vdash t : \tau \quad \Gamma \vdash \tau' : \mathcal{U} \quad \tau \leq \tau'}{\Gamma \vdash t : \tau'} \text{ (conv)}$$

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  - $=$ is defined in Coq's logic as an inductive predicate.
  - if $t \equiv t'$ and $t, t' : \tau$ then $t =_\tau t'$ is inhabited (has an element/proof).

Propositional equality

```
Inductive eq (A : Type) (x : A) : A -> Prop :=
| eq_refl : eq A x x.

Arguments eq {A}.
Arguments eq_refl {A x}, {A}.
(* we can write just `eq_refl' or `eq_refl y' *)

Notation "x = y :> A" := (@eq A x y) (at level 70).
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  eq_refl : forall (A : Type) (x : A), x = x
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- `eq` is a small propositional inductive type, so equality proofs may be eliminated to create programs.

Propositional equality elimination

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eq_ind =
fun (A : Type) (x : A) (P : A -> Prop) (t : P x)
    (y : A) (e : x = y) =>
match e in @eq _ _ y' return P y' with
| eq_refl => t
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  So `t` is required to have type `P x` inside the branch, which agrees with its actual type.
- The type of the entire match expression is `P y`.
- `eq_ind` computes on `eq_refl`:

  `eq_ind A a P t a (@eq_refl A a)` $\to_\iota$ `t`

Propositional equality elimination

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Used to implement type casts.

Symmetry of equality

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eq_sym =
fun (A : Type) (x y : A) (H : x = y) =>
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Transitivity of equality

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eq_trans =
fun (A : Type) (x y z : A) (H1 : x = y) (H2 : y = z) =>
match H2 in (_ = z') return (x = z') with
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Compatibility of functions with equality

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f_equal =
fun (A B : Type) (f : A -> B) (x y : A) (H : x = y) =>
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| eq_refl => @eq_refl B (f x)
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- E.g., if `H : a = b` and the goal is `P a` then `rewrite H` is `refine (eq_ind A b P _ a (eq_sym H)).`

Type casts

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- After all, `eq_refl` is the only constructor of `eq`, so we "should" have `p = eq_refl a`.
- This is indeed the case if `p` is closed (contains no free variables/axioms/opaque constants), because then `p` just computes to `eq_refl`.
- But in general it is not possible to prove `p = eq_refl a`!

Uniqueness of identity proofs

- The Uniqueness of Identity Proofs (UIP) axiom:

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Axiom UIP
: forall (A : Type) (x y : A) (p1 p2 : x = y), p1 = p2
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These axioms are equivalent. They are not provable in Coq's logic but consistent with it.

Invariance by substitution of reflexive equality proofs

```
Axiom eq_rect_eq
: forall (A : Type) (a : A) (P : A -> Type)
    (t : P a) (p : a = a),
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    (t : P a) (p : a = a),
    t = eq_rect A a P t a p.
```

- This axiom is equivalent to UIP.

Invariance by substitution of reflexive equality proofs

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- This axiom is the one actually present as an axiom in Coq's standard library, with UIP and UIP-refl derived from it as theorems.

Streicher's Axiom K

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Axiom K : forall (A : Type) (x : A) (P : x = x -> Type),
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- Compare the (definable) dependent eliminator for equality:

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- This rule holds definitionally in e.g. Agda, which makes working with dependent types a bit easier.
- Agda's dependent pattern matching relies on Streicher's K.
- Streicher's axiom K is incompatible with some recent developments in type theory (univalence, Homotopy Type Theory).

UIP for types with decidable equality

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```
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- In Coq, UIP is provable for types with decidable equality:

```
Theorem UIP_dec
: forall A : Type,
    (forall x y : A, {x = y} + {x <> y}) ->
    forall (x y : A) (p1 p2 : x = y), p1 = p2
```

Heterogeneous equality

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- Equality between two elements a, b of two different types A, B cannot be stated in terms of `eq`. Not even when A is propositionally equal to B!

Heterogeneous equality

```
vapp : forall {A n m},
         vector A n -> vector A m -> vector A (n + m).

Lemma lem_vapp_nil {A} :
  forall n (v : vector A n), vapp v vnil = v.
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```

```
Error:
In environment
A : Type
n : nat
v : vector A n
The term "v" has type "vector A n" while it
is expected to have type "vector A (n + 0)".
```

John Major equality

```
Inductive JMeq (A : Type) (x : A)
  : forall B : Type, B -> Prop :=
| JMeq_refl : JMeq A x A x.

Arguments JMeq [A] _ [B].
Arguments JMeq_refl {A x}, [A] _.

Notation "x ~= y" := (JMeq x y) (at level 70).
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- John Major equality enables us to state equality between elements in two different types.
- However, we may use John Major equality only when the two types are actually the same:

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JMeq_ind : forall (A : Type) (x : A) (P : A -> Prop),
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  JMeq_eq : forall (A : Type) (x y : A), x ~= y -> x = y
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- `JMeq_eq` is an axiom equivalent to UIP.

John Major equality

This works:

```
vapp : forall {A n m},
         vector A n -> vector A m -> vector A (n + m).

Lemma lem_vapp_nil {A} :
  forall n (v : vector A n), vapp v vnil ~= v.
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John Major equality

John Major's "classless society" widened people's aspirations to equality, but also the gap between rich and poor. (...) In much the same way, `JMeq` forms equations between members of any type, but they cannot be treated as equals (i.e. substituted) unless they are of the same type. Just as before, each thing is only equal to itself.

Conor McBride, "Dependently Typed Functional Programs and their Proofs", PhD thesis, 1999

Dependent equality

```
Inductive eq_dep (U : Type) (P : U -> Type) (p : U) (x : P p)
  : forall q : U, P q -> Prop :=
| eq_dep_intro : eq_dep U P p x p x

eq_dep_ind =
fun (U : Type) (P : U -> Type) (p : U) (x : P p)
  (Q : forall q : U, P q -> Prop) (f : Q p x) (q : U)
  (y : P q) (e : eq_dep U P p x q y) =>
match e in eq_dep _ _ _ _ q' y' return Q q' y' with
| eq_dep_intro _ _ _ _ => f
end
: forall (U : Type) (P : U -> Type) (p : U) (x : P p)
    (Q : forall q : U, P q -> Prop),
    Q p x -> forall (q : U) (y : P q),
    eq_dep U P p x q y -> Q q y
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The eliminator `eq_dep_ind` does not depend on any axioms. We may rewrite dependent equalities without UIP.

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```

To convert `eq_dep` to `eq` we need the axiom

```
eq_dep_eq
: forall (U : Type) (P : U -> Type) (p : U) (x y : P p),
    eq_dep U P p x p y -> x = y
```

which is equivalent to UIP.

Dependent equality

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- `JMeq` is equivalent to `eq_dep Type (fun X => X)`.

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```

- `JMeq` is equivalent to `eq_dep Type (fun X => X)`.
- `eq_dep` is strictly finer than `JMeq`:

```
forall U P p q (x : P p) (y : P q),
  eq_dep U P p x q y -> x ~= y.

exists U P p q (x : P p) (y : P q),
  x ~= y /\ ~ eq_dep U P p x q y.
```